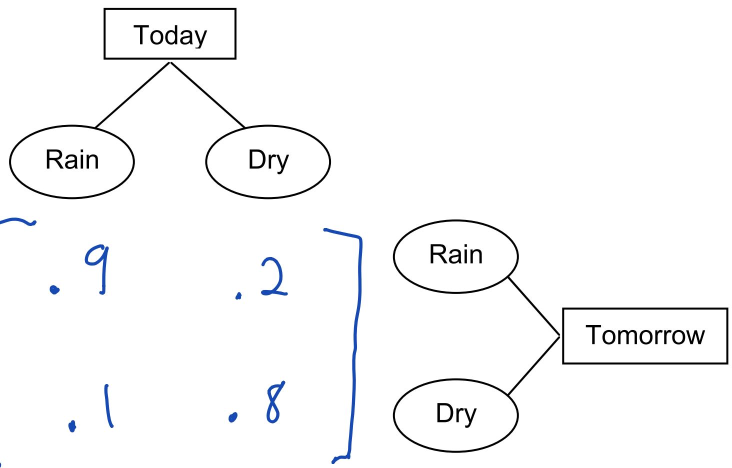
Example 13.1
It rains a lot in Gloomtown. If it rains today, there is a 90% probability that it will rain tomorrow. If it is dry today, there is an 80% chance it will be dry tomorrow. What percentage of days does it rain in Gloomtown?

a. Let's start by coming up with a transition matrix for this Markov Chain.



$$\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}$$

b. Let's familiarize ourselves with the process. What does a state vector represent in the context of this problem?

c. Let's establish that if it rains today, then the state vector for tomorrow is $P^1[1,0]^T$, the state vector two days from now is $P^2[1,0]^T$ and the state vector 3 days from now is $P^3[1,0]^T$

d. If it rains today, what are the state vectors for each of the next three days? What about if it's dry today? Then project into the future … if it rains today, what are the state vectors 50 days from now, 51 days from now, and 52 days from now? What about if it's dry today? What is the *steady state* vector for this Markov Chain? What does it tell us in the context of this problem?

e. Let's use eigenvalues to determine the steady state vector for Gloomtown's rainy situation.

Example 13.2
Pretend that Manhattan only has Midtown, the Upper East Side, and the Upper West Side. Suppose that cabbie shifts change only at 6 am and 6 pm. Suppose that a state vector for the distribution of cabs has form $[\text{Midtown}, \text{UES}, \text{UWS}]^T$ and that the transition matrix over each 5 minute interval between 6 am and 6 pm is the matrix P. Find the steady state vector for this model. Suppose that the cabs are evenly distributed at the start of the 6 am shift; how well does the steady state vector predict the distribution of the cabs at the end of that shift?

$$P = \begin{bmatrix} .5 & .3 & .4 \\ .2 & .6 & .1 \\ .3 & .1 & .5 \end{bmatrix}$$

Ex 13.1

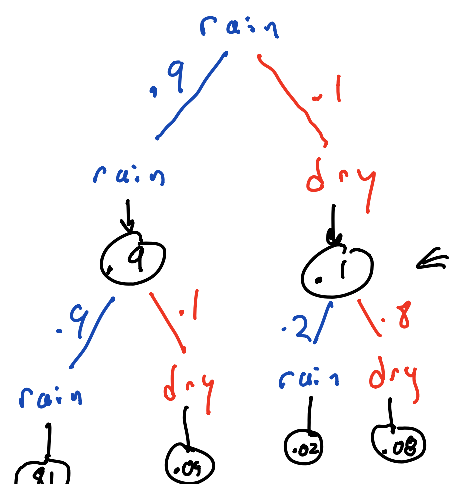
Transition Matrix $P = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}$.

b. Our state vectors have form

$$\begin{bmatrix} \text{Probability that it rains on day } k \\ \text{Probability that it is dry on day } k \end{bmatrix}$$

today is day 0.

c. It rained today, so $\vec{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



probabilities for tomorrow

Probabilities for two days from now and they add to 1 (100 %)

4.9 : 1-13 odd

$$\begin{aligned}
\vec{v}_2 &= P^2 \vec{v}_0 \\
&= P \cdot P\vec{v}_0 \\
&= P \cdot \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= P \cdot \begin{bmatrix} .9 \\ .1 \end{bmatrix} \quad \text{This is } \vec{v}_1 \\
&= \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \cdot \begin{bmatrix} .9 \\ .1 \end{bmatrix} \\
&= \begin{bmatrix} (.9)(.9) + (.1)(.2) \\ (.9)(.1) + (.1)(.8) \end{bmatrix} \\
&= \begin{bmatrix} .83 \\ .17 \end{bmatrix}
\end{aligned}$$

d) rain today

$\vec{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} .9 \\ .1 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} .83 \\ .17 \end{bmatrix}$

$$\vec{v}_3 = P\vec{v}_2 = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}\begin{bmatrix} .83 \\ .17 \end{bmatrix} = \begin{bmatrix} .781 \\ .219 \end{bmatrix}$$

dry today

$\vec{v}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} .2 \\ .8 \end{bmatrix}$

$$\vec{v}_2 = P\vec{v}_1 = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}\begin{bmatrix} .2 \\ .8 \end{bmatrix} = \begin{bmatrix} .34 \\ .66 \end{bmatrix}$$

$$\vec{v}_3 = P\vec{v}_2 = \begin{bmatrix} .438 \\ .562 \end{bmatrix}$$

d. Check this out ...

If it's rainy today, 50 days hence the state vector is:

$$\vec{v}_{50} = P^{50}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \approx \begin{bmatrix} .666666667 \\ .33333333 \end{bmatrix}$$

If it's dry today

$$\vec{v}_{50} = P^{50}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} .666666665 \\ .333333335 \end{bmatrix}$$

Gee willikers, regardless of the state today, 50 days from now it's 66.7% likely it rains and 33.3% likely it is dry.

e) Let's prove this!! :)

Let's diagonalize P.

Characteristic eq: $\det(P - \lambda I) = 0$

$$\begin{vmatrix} .9-\lambda & .2 \\ .1 & .8-\lambda \end{vmatrix} = 0 \Rightarrow (.9-\lambda)(.8-\lambda) - .02 = 0$$

$$\Rightarrow \lambda^2 - 1.7\lambda + .7 = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - .7) = 0$$

Field trip! 1 is always an eigenvalue of Stochastic Matrix ... that's what creates the steady state vector.

$$P\vec{x} = \vec{x}$$

Steady state vector

Proof for 2x2 P matrix

$$P = \begin{bmatrix} p & q \\ 1-p & 1-q \end{bmatrix}$$

Consider $P\vec{x} = 1\vec{x}$

$$\begin{bmatrix} p & q \\ 1-p & 1-q \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} p-1 & q & 0 \\ 1-p & 1-q-1 & 0 \end{array}\right] \xrightarrow{R_1+R_2 \to R_2} \left[\begin{array}{cc|c} p-1 & q & 0 \\ 0 & 0 & 0 \end{array}\right]$$

QED

Back to task

$\underline{\lambda = 1}$

$$\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -.1 & .2 & 0 \\ .1 & -.2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Gen Sol: $\begin{cases} x_1 = 2x_2 \\ x_2 \text{ is free} \end{cases}$ Basis: $\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

$\underline{\lambda = .7}$

$$\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .7x_1 \\ .7x_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} .2 & .2 & 0 \\ .1 & .1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Gen Sol: $\begin{cases} x_1 = -x_2 \\ x_2 \text{ is free} \end{cases}$ Basis: $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

$\therefore P = QDQ^{-1}$ where

$$Q = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & .7 \end{bmatrix}, Q^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

$$\therefore P^k \vec{v}_0 = Q D^k Q^{-1} \vec{v}_0$$

$$= \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1^k & 0 \\ 0 & .7^k \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} v_{01} \\ v_{02} \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & -(.7^k) \\ 1 & .7^k \end{bmatrix}\begin{bmatrix} v_{01} + v_{02} \\ -v_{01} + 2v_{02} \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2(v_{01}+v_{02}) - (.7^k)(-v_{01}+2v_{02}) \\ 1(v_{01}+v_{02}) + (.7^k)(-v_{01}+2v_{02}) \end{bmatrix}$$

$$\therefore \lim_{k\to\infty} P^k \vec{v}_0 = \frac{1}{3}\begin{bmatrix} 2(v_{01}+v_{02}) - 0 \\ 1(v_{01}+v_{02}) + 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3}(v_{01}+v_{02}) \\ \frac{1}{3}(v_{01}+v_{02}) \end{bmatrix}$$

regardless of the the initial state, $v_{01} + v_{02} = 1$

$\therefore$ regardless of the initial state, the steady state stochastic vector is $\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$

Ex 13.2

where cab is now

$$P = \begin{array}{c} \begin{array}{ccc} MT & UES & UWS \end{array} \\ \begin{bmatrix} .5 & .3 & .4 \\ .2 & .6 & .1 \\ .3 & .1 & .5 \end{bmatrix} \end{array} \begin{array}{l} MT \\ UES \\ UWS \end{array}$$

where cab is in 5 minutes

for example, $p_{32} = .1$ tells us that if the cab is on the upper east side, there is a 10% chance that five minutes hence it will be on the upper west side.

b. The steady (stochastic) vector is an eigen vector associated with $\lambda = 1$

$$\begin{bmatrix} .5 & .3 & .4 \\ .2 & .6 & .1 \\ .3 & .1 & .5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} -.5 & .3 & .4 & | & 0 \\ .2 & -.4 & .1 & | & 0 \\ .3 & .1 & -.5 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -19/14 & | & 0 \\ 0 & 1 & -13/14 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Gen Solution: $\begin{cases} x_1 = \frac{19}{14}x_3 \\ x_2 = \frac{13}{14}x_3 \\ x_3 \text{ is free} \end{cases}$

A 1-eigenspace basis is $\left\{ \begin{bmatrix} 19 \\ 13 \\ 14 \end{bmatrix} \right\}$

$\therefore$ The stochastic steady state vector is:

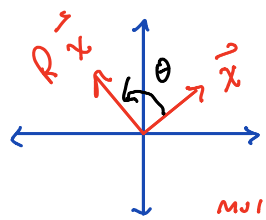
$$\frac{1}{19+13+14}\begin{bmatrix} 19 \\ 13 \\ 14 \end{bmatrix} = \begin{bmatrix} 19/46 \\ 13/46 \\ 14/46 \end{bmatrix} \approx \begin{bmatrix} 41\% \\ 28\% \\ 30\% \end{bmatrix}$$

Shifts: 6am to 6pm

"States" occur every five minutes, so there are 144 "states" after $\vec{v}_0$.

$$\vec{v}_{144} = P^{144}\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \approx \begin{bmatrix} .41 \\ .28 \\ .30 \end{bmatrix}$$

13.4 //



$R\vec{x} \neq \lambda\vec{x}$ for all $\lambda \in \mathbb{R}$

Multiplying by λ changes length, not direction (other than 180°)

QED

13.3: The range of T is $\left\{ c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ -1 \end{bmatrix} \middle| c_1, c_2 \in \mathbb{R} \right\}$,

so a basis for the range of T is clearly $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

The general solution to $A\vec{x} = \vec{0}$ is $\begin{cases} x_1 = x_2 \\ x_2 \text{ is free} \end{cases}$,

so $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is also a basis for the kernel of T. Thus, $\forall \vec{x} \in \mathbb{R}^2$

$$\underset{\substack{\text{range} \\ \text{spanned by} \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix}}}{\vec{x} \xrightarrow{T} k\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \xrightarrow[\substack{\text{kernel} \\ \text{contains all} \\ k\begin{bmatrix} 1 \\ 1 \end{bmatrix}}]{} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \forall \vec{x} \in \mathbb{R}^2 \quad A \cdot A\vec{x} = \vec{0}$

$\therefore \quad AA \cdot \vec{x} = \vec{0} \quad \forall \vec{x} \in \mathbb{R}^2$

$\therefore \quad A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Ex 13.5)

$$A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 1 & 5 & -2 \\ -2 & 2 & -8 & 0 \\ 3 & 2 & 7 & 10 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b. $\operatorname{col}(A^T)^{\perp} = \operatorname{row}(A)^{\perp}$

$= \operatorname{nul}(A)^{\cancel{A}}$

Suppose that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \operatorname{row}(A)^{\perp}$

Then $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$

So $x_1 + 2x_2 + x_3 + 4x_4 = 0$

likewise $\begin{bmatrix} 2 \\ 1 \\ 5 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$

So $2x_1 + x_2 + 5x_3 - 2x_4 = 0$

$\vdots$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 1 & 5 & -2 \\ -2 & 2 & -8 & 0 \\ 3 & 2 & 7 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

any vector in nul(A) is orthogonal to any vector in the row space of A.

Back to the problem...

$\operatorname{col}(A^T)^{\perp} = \operatorname{row}(A)^{\perp}$

$= \operatorname{nul}(A)$

$\therefore$ from RREF(A) a basis for $\operatorname{col}(A^T)^{\perp}$ is $\left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ ✓

Gen sol: $\begin{cases} x_1 = -3x_3 \\ x_2 = x_3 \\ x_3 \text{ is free}, \; x_4 = 0 \end{cases}$

a) $\text{null}(A^T)^{\perp} = \text{row}(A^T)$

$= \text{col}(A)$

$\therefore$ A basis for $\text{null}(A^T)^{\perp}$ is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \\ 10 \end{bmatrix} \right\}.$$

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side note

A basis for row(A) is

$$\{ [1, 0, 3, 0], [0, 1, -1, 0], [0, 0, 0, 1] \}$$

4^{th} row of A

$$[3, 2, 7, 10] = \underset{\substack{\downarrow \\ 3}}{c_1} [1, 0, 3, 0] + \underset{\substack{\downarrow \\ 2}}{c_2} [0, 1, -1, 0] + \underset{\substack{\downarrow \\ 10}}{c_3} [0, 0, 0, 1]$$