Stochastic Vectors, Stochastic Matrices, and Markov Chains

A ***stochastic vector*** is vector containing no negative components and whose components sum to 1.

A ***stochastic matrix*** is a matrix whose column vectors are stochastic vectors.

Suppose that any given member of a set must be in exactly 1 of a finite number of states at any given time. Suppose further that the state of each member is noted over regular time intervals. Then the stochastic vector $\vec{v}_k = \left[q_1^{(k)}, q_2^{(k)}, \ldots, q_n^{(k)} \right]$ where $q_i^{(k)}$ is the probability that a given member of the set is in state i at time k is called a ***state vector***

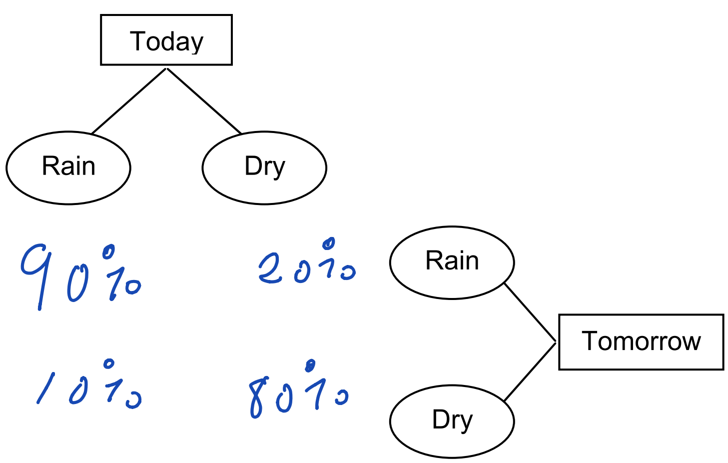
Suppose that p_{ij} is the probability that a member in state j will transition to state i the next time an account is made. Suppose further that these probabilities do not change over time. Then the sequence $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots$ is called a ***Markov Chain***.

Furthermore, $\vec{v}_{k+1} = P\vec{v}_k$ where $P = \left[p_{ij} \right]_{n \times n}$ and P is called the ***transition matrix*** for the Markov Chain.

Example 11.1

It rains a lot in Gloomtown. If it rains today, there is a 90% probability that it will rain tomorrow. If it is dry today, there is an 80% chance it will be dry tomorrow. What percentage of days does it rain in Gloomtown?

a. Let's start by coming up with a transition matrix for this Markov Chain.

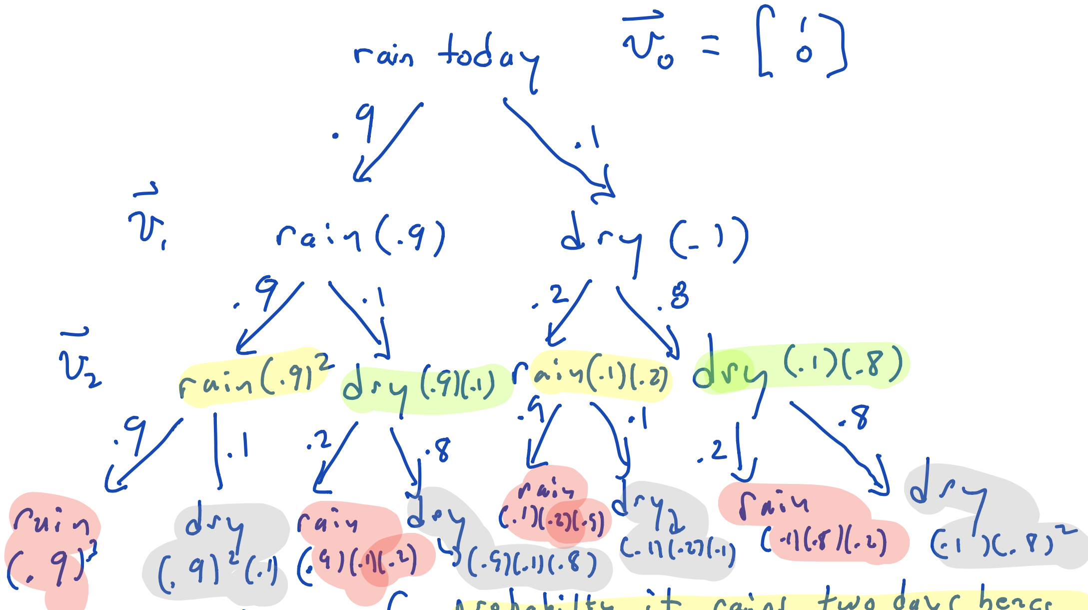


Transition matrix: $P = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}$

b. Let's familiarize ourselves with the process. What does a state vector represent in the context of this problem?

$$v_K = \begin{bmatrix} \text{probability that day } K \text{ is rainy} \\ \text{probability that day } K \text{ is dry} \end{bmatrix}$$

c. Let's establish that if it rains today, then the state vector for tomorrow is $P^1[1,0]^T$, the state vector two days from now is $P^2[1,0]^T$ and the state vector 3 days from now is $P^3[1,0]^T$



$$\vec{v}_2 = \begin{bmatrix} \text{probability it rains two days hence} \\ \text{probability it's dry two days hence} \end{bmatrix}$$

$$= \begin{bmatrix} (.9)(.9) + (.2)(.1) \\ (.1)(.9) + (.8)(.1) \end{bmatrix}$$

$$= \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} .9 \\ .1 \end{bmatrix}$$

$$= P\vec{v}_1$$

$$\vec{v}_1 = \begin{bmatrix} .9 \\ .1 \end{bmatrix}$$

$$= \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= P\vec{v}_0$$

$\therefore$ $\vec{v}_2 = P\vec{v}_1$

$= P \cdot P\vec{v}_0$

$= P^2\vec{v}_0$

In general, $\vec{v}_K = P^K\vec{v}_0$

$$\vec{v}_3 = \begin{bmatrix} \text{probability it rains three days hence} \\ \text{probability it is dry three days hence} \end{bmatrix}$$

$$= \begin{bmatrix} (.9)^3 + (.9)(.1)(.2) + (.1)(.2)(.9) + (.1)(.8)(.2) \\ (.9)^2(.1) + (.9)(.1)(.8) + (.1)(.2)(.1) + (.1)(.8)^2 \end{bmatrix}$$

$$= \begin{bmatrix} .781 \\ .219 \end{bmatrix}$$ (adds to one ☺)

$$\begin{aligned} \vec{v}_3 &= P\vec{v}_2 \\ &= P \cdot P\vec{v}_1 \\ &= P \cdot P \cdot P\vec{v}_0 \\ &= P^3\vec{v}_0 \\ &= \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} .781 \\ .219 \end{bmatrix} \end{aligned}$$

☺

d. If it rains today, what are the state vectors for each of the next three days? What about if it's dry today? Then project into the future … if it rains today, what are the state vectors 50 days from now, 51 days from now, and 52 days from now? What about if it's dry today? What is the *steady state* vector for this Markov Chain? What does it tell us in the context of this problem?

rain today $\left(\vec{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

$$\vec{v}_1 = \begin{bmatrix} .9 \\ .1 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} .83 \\ .17 \end{bmatrix}$$

$$\vec{v}_3 = P\vec{v}_2 = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}\begin{bmatrix} .83 \\ .17 \end{bmatrix} = \begin{bmatrix} .781 \\ .219 \end{bmatrix}$$

$\vdots$

$$\vec{v}_{50} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}^{50}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \approx \begin{bmatrix} .666667 \\ .333333 \end{bmatrix}$$

dry today $\left(\vec{v}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

$$\vec{v}_1 = \begin{bmatrix} .2 \\ .8 \end{bmatrix}$$

$$\vec{v}_2 = P\vec{v}_1 = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}\begin{bmatrix} .2 \\ .8 \end{bmatrix} = \begin{bmatrix} .34 \\ .66 \end{bmatrix}$$

$$\vec{v}_3 = P\vec{v}_2 = \begin{bmatrix} .438 \\ .562 \end{bmatrix}$$

$\vdots$

$$\vec{v}_{50} = \begin{bmatrix} .9 & .2 \\ .1 & .5 \end{bmatrix}^{50}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \approx \begin{bmatrix} .666667 \\ .333333 \end{bmatrix}$$

In the long term, the initial state vector serves no probative value. The odds 50 days hence are independent of what happens in state 0.

e. Let's use eigenvalues to determine the steady state vector for Gloomtown's rainy situation.

The steady state vector is the stochastic vector that satisfies $P\vec{x} = \vec{x}$. (Note that this implies that 1 is an eigenvalue of P.) (calc check: $\lambda = 1$, $\lambda = .7$)

$$P\vec{x} = \vec{x} \Rightarrow \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1x_1 \\ 1x_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -.1 & .2 & 0 \\ .1 & -.2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Gen Sol: $\begin{cases} x_1 = 2x_2 \\ x_2 \text{ is free} \end{cases}$: Basis: $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

$\therefore$ The steady state vector is:

$$\frac{1}{2+1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$ woo hoo!

Example 11.5
Demonstrate that 1 is always an eigenvalue for a 2 x 2 stochastic matrix.

$$\begin{bmatrix} p & 1-q \\ 1-p & q \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{cases} px_1 + (1-q)x_2 = x_1 \\ (1-p)x_1 + qx_2 = x_2 \end{cases} \Rightarrow \begin{cases} (p-1)x_1 + (1-q)x_2 = 0 \\ (1-p)x_1 + (q-1)x_2 = 0 \end{cases}$$

$$\left[\begin{array}{cc|c} p-1 & 1-q & 0 \\ 1-p & q-1 & 0 \end{array}\right] \xrightarrow{R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} p-1 & 1-q & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore$ The vector equation has non-trivial solutions

$\therefore$ 1 is indeed an eigenvalue.

Example 11.2

Pretend that Manhattan only has Midtown, the Upper East Side, and the Upper West Side. Suppose that cabbie shifts change only at 6 am and 6 pm. Suppose that a state vector for the distribution of cabs has form $[\text{Midtown}, \text{UES}, \text{UWS}]^T$ and that the transition matrix over each 5 minute interval between 6 am and 6 pm is the matrix P. Find the steady state vector for this model. Suppose that the cabs are evenly distributed at the start of the 6 am shift; how well does the steady state vector predict the distribution of the cabs at the end of that shift?

where the cab is

$$P = \begin{bmatrix} .5 & .3 & .4 \\ .2 & .6 & .1 \\ .3 & .1 & .5 \end{bmatrix}$$

where the cab will be 5 minutes later.

Midtown

Background

If a cab is in midtown at, say, 7:25 am, then at 7:30 am there is a 50% probability that it will still be in midtown, 20% probality that it's on the UES, and 30% UWS.

to the question...

The steady state vector is the stochastic vector in the 1-eigenspace.

$$\begin{bmatrix} .5 & .3 & .4 \\ .2 & .6 & .1 \\ .3 & .1 & .5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -.5 & .3 & .4 & 0 \\ .2 & -.4 & .1 & 0 \\ .3 & .1 & -.5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -19/14 & 0 \\ 0 & 1 & -13/14 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Gen Solution: $\begin{cases} x_1 = \frac{19}{14}x_3 \\ x_2 = \frac{13}{14}x_3 \\ x_3 \text{ is free.} \end{cases}$ $\therefore$ A 1-eigenvector is $\begin{bmatrix} 19 \\ 13 \\ 14 \end{bmatrix}$

and the stochastic steady state is $\begin{bmatrix} 19/46 \\ 13/46 \\ 7/23 \end{bmatrix}$

$(19 + 13 + 14 = 46)$

onward

Between 6am & 6pm, there are 144 "events" (which occur every five minutes). So if the cabs are evenly distributed at 6am, then at 6pm:

$$\vec{v}_{144} = P^{144}\vec{v}_0$$

$$= \begin{bmatrix} .5 & .3 & .4 \\ .2 & .6 & .1 \\ .3 & .1 & .5 \end{bmatrix}^{144} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$\approx \begin{bmatrix} .41304 \\ .28260 \\ .30434 \end{bmatrix}$$

Steady State: $\begin{bmatrix} 19/46 \\ 13/46 \\ 7/23 \end{bmatrix} \approx \begin{bmatrix} .41304 \\ .28260 \\ .30430 \end{bmatrix}$

$\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ span $\left\{ \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right\}$

Example 11.4

What are bases for the kernel and range of the linear transformation $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

What does this imply must be true about A^2.

The kernel of T are solutions to:

$T(\vec{x}) = \vec{0}$, so $A\vec{x} = \vec{0}$

$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ Basis for the Kernel of T is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

The range of T is the set of vectors that can be expressed as $x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

i.e. the range of T is the column space of A.

obviously, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ spans this space all on its own, so a basis for the range of T is also $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

$$\vec{x} \xrightarrow[\substack{\text{range} \\ \text{is} \\ \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}}]{T} \begin{bmatrix} k \\ k \end{bmatrix} \xrightarrow[\substack{\text{null} \\ \text{space} \\ \text{is} \\ \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}}]{T} \vec{0}$$

$\therefore \quad T(T(\vec{x})) = \vec{0} \quad \forall \vec{x} \in \mathbb{R}^2$

$\therefore \quad A(A\vec{x}) = \vec{0} \quad \forall \vec{x} \in \mathbb{R}^2$

$\therefore \quad A^2 \vec{x} = \vec{0} \quad \forall \vec{x} \in \mathbb{R}^2$

$\therefore \quad A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$