The general solution to the Differential Equation $\frac{dy}{dt} = \lambda y$ is $y = Ce^{\lambda t}$.

note: $y = Ce^{\lambda t} \Rightarrow \frac{dy}{dt} = Ce^{\lambda t} \cdot \lambda$

$\Rightarrow \frac{dy}{dt} = \lambda y$ ✓

Example 11.1 : $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$$\frac{d\vec{y}}{dt} = A\vec{y} \Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} y_1' = 2y_1 \\ y_2' = 3y_2 \end{cases}$$

$$\Rightarrow \begin{cases} y_1 = c_1 e^{2t} \\ y_2 = c_2 e^{3t} \end{cases} \text{(general Solution)}$$

Initial Condition

$$\vec{y}(0) = \begin{bmatrix} -4 \\ -2 \end{bmatrix} \Rightarrow \begin{cases} -4 = c_1 e^0 \\ -2 = c_2 e^0 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = -4 \\ c_2 = -2 \end{cases}$$

$\therefore$ The specific solution to $\vec{y}' = A\vec{y}$ is $\vec{y} = \begin{bmatrix} -4e^{2t} \\ -2e^{3t} \end{bmatrix}$

Check: $\vec{y}' = \begin{bmatrix} -8e^{2t} \\ -6e^{3t} \end{bmatrix}$

$$A\vec{y} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -4e^{2t} \\ -2e^{3t} \end{bmatrix} = \begin{bmatrix} -8e^{2t} \\ -6e^{3t} \end{bmatrix} \checkmark$$

Ex 11.2 $\begin{cases} y_1' = y_1 + 2y_2 \\ y_2' = -y_1 + 4y_2 \end{cases}$

This system can be expressed thus:

$$\vec{y}' = A\vec{y} \text{ where } A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$$

Characteristic Eq: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(4-\lambda) + 2 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 3) = 0$$

$\therefore$ Eigenvalues are 2 and 3

$\lambda = 2$

$$\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ -1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Gen sol: $\begin{cases} x_1 = 2x_2 \\ x_2 \text{ is free} \end{cases}$

Basis: $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

$\lambda = 3$

$$\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Gen sol: $\begin{cases} x_1 = x_2 \\ x_2 \text{ is free} \end{cases}$

Basis: $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$\therefore \quad A = PDP^{-1}$ where $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$,

$$P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Define $\vec{w} = P^{-1}\vec{y}$ so that $\vec{y} = P\vec{w}$ and $\vec{y}\,' = P\vec{w}\,'$

$$\vec{y}\,' = A\vec{y} \implies P\vec{w}\,' = A \cdot P\vec{w}$$

$$\implies P\vec{w}\,' = PDP^{-1} \cdot P\vec{w}$$

Straight passage henceforth

$$\implies P^{-1}P\vec{w}\,' = P^{-1}PDP^{-1}P\vec{w}$$

$$\implies \vec{w}\,' = D\vec{w}$$ (The system has been decoupled)

Reboot: $\quad \vec{y}\,' = A\vec{y} \implies \vec{w}\,' = D\vec{w}$

$$\implies \begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\implies \begin{cases} w_1' = 2w_1 \\ w_2' = 3w_2 \end{cases}$$

$$\implies \begin{cases} w_1 = C_1 e^{2t} \\ w_2 = C_2 e^{3t} \end{cases}$$

$$\vec{y} = P\vec{w} \implies \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 e^{2t} \\ C_2 e^{3t} \end{bmatrix}$$

$$\implies \begin{cases} y_1 = 2C_1 e^{2t} + C_2 e^{3t} \\ y_2 = C_1 e^{2t} + C_2 e^{3t} \end{cases}$$ (general Solution)

$$\vec{y}(0) = \begin{bmatrix} -5 \\ 0 \end{bmatrix} \implies \begin{cases} 2C_1 + C_2 = -5 \\ C_1 + C_2 = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} C_1 = -5 \\ C_2 = 5 \end{cases}$$

$\therefore$ The specific solution to $\vec{y}\,' = A\vec{y}$ where $\vec{y}(0) = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$ is:

$$\begin{cases} y_1 = -10e^{2t} + 5e^{3t} \\ y_2 = -5e^{2t} + 5e^{3t} \end{cases}$$

Check

$$\vec{y} = \begin{bmatrix} -10e^{2t} + 5e^{3t} \\ -5e^{2t} + 5e^{3t} \end{bmatrix} \Longrightarrow \vec{y}\,' = \begin{bmatrix} -20e^{2t} + 15e^{3t} \\ -10e^{2t} + 15e^{3t} \end{bmatrix}$$

$$A\vec{y} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} -10e^{2t} + 5e^{3t} \\ -5e^{2t} + 5e^{3t} \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot(-10e^{2t} + 5e^{3t}) + 2\cdot(-5e^{2t} + 5e^{3t}) \\ -1\cdot(-10e^{2t} + 5e^{3t}) + 4\cdot(-5e^{2t} + 5e^{3t}) \end{bmatrix}$$

$$= \begin{bmatrix} -20e^{2t} + 15e^{3t} \\ -10e^{2t} + 15e^{3t} \end{bmatrix}$$ Look at that!

Example 3: $\vec{y}\,' = A\vec{y}$ where $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

and $\vec{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Characteristic Eq. of A: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0 \Longrightarrow (1-\lambda)(4-\lambda) - 4 = 0$$

$$\Longrightarrow \lambda^2 - 5\lambda = 0$$

$$\Longrightarrow \lambda(\lambda - 5) = 0$$

$\therefore$ The eigenvalues are 0 and 5

$\underline{\lambda = 0}$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

gen sol: $\begin{cases} x_1 = -2x_2 \\ x_2 \text{ is free} \end{cases}$

Basis: $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$

$\underline{\lambda = 5}$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5x_1 \\ 5x_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

gen sol: $\begin{cases} x_1 = \frac{1}{2}x_2 \\ x_2 \text{ is free} \end{cases}$

Basis: $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

$\therefore$ $A = PDP^{-1}$ where $P = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$,

$$P^{-1} = -\frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}$$

Define $\vec{w} = P^{-1}\vec{y}$ so that $\vec{y} = P\vec{w}$ and $\vec{y}' = P\vec{w}'$

$$\vec{y}' = A\vec{y} \implies \vec{w}' = D\vec{w}$$

$$\implies \begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\implies \begin{cases} w_1' = 0w_1 \\ w_2' = 5w_2 \end{cases}$$

$$\implies \begin{cases} w_1 = C_1 \\ w_2 = C_2 e^{5t} \end{cases}$$

$\vec{y} = P\vec{w}$

$$\implies \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 e^{5t} \end{bmatrix}$$

$$\Longrightarrow \begin{cases} y_1 = -2c_1 + c_2 e^{5t} \\ y_2 = c_1 + 2c_2 e^{5t} \end{cases}$$

$$\vec{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Longrightarrow \begin{cases} -2c_1 + c_2 = 1 \\ c_1 + 2c_2 = -1 \end{cases}$$

$$\Longrightarrow \begin{cases} c_1 = -3/5 \\ c_2 = -1/5 \end{cases}$$

$\therefore$ The specific solution to $\vec{y}\,' = A\vec{y}$ where $\vec{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is:

$$\begin{cases} y_1 = \frac{6}{5} - \frac{1}{5} e^{5t} \\ y_2 = -\frac{3}{5} - \frac{2}{5} e^{5t} \end{cases}$$

Check

$$\vec{y} = \begin{bmatrix} \frac{6}{5} - \frac{1}{5} e^{5t} \\ -\frac{3}{5} - \frac{2}{5} e^{5t} \end{bmatrix} \Longrightarrow \vec{y}\,' = \begin{bmatrix} -e^{5t} \\ -2e^{5t} \end{bmatrix}$$

$$A\vec{y} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \frac{6}{5} - \frac{1}{5} e^{5t} \\ -\frac{3}{5} - \frac{2}{5} e^{5t} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot \left(\frac{6}{5} - \frac{1}{5} e^{5t}\right) + 2 \cdot \left(-\frac{3}{5} - \frac{2}{5} e^{5t}\right) \\ 2 \cdot \left(\frac{6}{5} - \frac{1}{5} e^{5t}\right) + 4 \cdot \left(-\frac{3}{5} - \frac{2}{5} e^{5t}\right) \end{bmatrix}$$

$$= \begin{bmatrix} -e^{5t} \\ -2e^{5t} \end{bmatrix}$$ Groovy!

Ex 11.4 // $\vec{y}' = A\vec{y}$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Characteristic eq: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \implies \lambda^2 + 1 = 0$$
$$\implies \lambda^2 = -1$$
$$\implies \lambda = \pm i$$

$\underline{\lambda = i}$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} i x_1 \\ i x_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Gen Sol: $\begin{cases} x_1 = -i x_2 \\ x_2 \text{ is free} \end{cases}$

Basis: $\left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$

$\underline{\lambda = -i}$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -i x_1 \\ -i x_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Gen Sol: $\begin{cases} x_1 = i x_2 \\ x_2 \text{ is free} \end{cases}$

Basis: $\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$

$\therefore$ $A = PDP^{-1}$ where $P = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$ and

$D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, P^{-1} = something.

start point on test

Define $\vec{w} = P^{-1}\vec{y}$ so that $\vec{y} = P\vec{w}$ and $\vec{y}' = P\vec{w}'$

$$\vec{y}' = A\vec{y} \Rightarrow \vec{w}' = D\vec{w}$$

$$\Rightarrow \begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} w_1' = i w_1 \\ w_2' = -i w_2 \end{cases}$$

$$\Rightarrow \begin{cases} w_1 = C_1 e^{it} \\ w_2 = C_2 e^{-it} \end{cases}$$

$$e^{\theta i} = \cos(\theta) + i\sin(\theta)$$

$$\Rightarrow \begin{cases} w_1 = C_1(\cos(t) + i\sin(t)) \\ w_2 = C_2(\cos(-t) + i\sin(-t) \end{cases}$$

$$\Rightarrow \begin{cases} w_1 = C_1\cos(t) + C_1 i\sin(t) \\ w_2 = C_2\cos(t) - C_2 i\sin(t) \end{cases}$$

$$\vec{y} = P\vec{w} \Rightarrow \vec{y} = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (C_1\cos(t) + C_1 i\sin(t)) \\ (C_2\cos(t) - C_2 i\sin(t)) \end{bmatrix}$$

$$\Rightarrow \begin{cases} y_1 = -iC_1\cos(t) + C_1\sin(t) + iC_2\cos(t) + C_2\sin(t) \\ y_2 = \boxed{C_1\cos(t) + iC_1\sin(t) + C_2\cos(t) - iC_2\sin(t)} \end{cases}$$

(note: $\cos(0) = 1$

$\sin(0) = 0$

$$y(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \Rightarrow \begin{cases} -ic_1 + 0 + ic_2 + 0 = 3 \\ c_1 + 0 + c_2 - 0 = 3 \end{cases}$$

$$\left[\begin{array}{cc|c} -i & i & 3 \\ 1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & \frac{3}{2} + \frac{3}{2}i \\ 0 & 1 & \frac{3}{2} - \frac{3}{2}i \end{array}\right]$$

$$\therefore \begin{cases} c_1 = \frac{3}{2} + \frac{3}{2}i \\ c_2 = \frac{3}{2} - \frac{3}{2}i \end{cases}$$

$\therefore$

$$y_1 = -i\left(\frac{3}{2} + \frac{3}{2}i\right)\cos(t) + \left(\frac{3}{2} + \frac{3}{2}i\right)\sin(t) + i\left(\frac{3}{2} - \frac{3}{2}i\right)\cos(t) + \left(\frac{3}{2} - \frac{3}{2}i\right)\sin(t)$$

$$= \cancel{-\frac{3}{2}i\cos(t)} + \frac{3}{2}\cos(t) + \frac{3}{2}\sin(t) + \cancel{\frac{3}{2}i\sin(t)} + \cancel{\frac{3}{2}i\cos(t)} + \frac{3}{2}\cos(t) + \frac{3}{2}\sin(t) - \cancel{\frac{3}{2}i\sin(t)}$$

$$= 3\cos(t) + 3\sin(t)$$

$$y_2 = -3\sin(t) + 3\cos(t)$$

$$\vec{y} = \begin{bmatrix} 3\cos(t) + 3\sin(t) \\ -3\sin(t) + 3\cos(t) \end{bmatrix} \Rightarrow \vec{y}\,' = \begin{bmatrix} -3\sin(t) + 3\cos(t) \\ -3\cos(t) - 3\sin(t) \end{bmatrix}$$

$$= \begin{bmatrix} y_2 \\ -y_1 \end{bmatrix}$$