Definitions 10.1-10.3: Eigenvalues and Eigenvectors (of square matrices)

A non-zero vector $\vec{v}$ is called ***an eigenvector*** of the square matrix A if there exists a scalar, λ, with the property that $A\vec{v} = \lambda\vec{v}$. If such a vector and scalar exist, the scalar λ is called ***an eigenvalue*** of A.

The eigenvalues of A are the solutions to the equation $\det(A - \lambda I) = 0$; this equation is called ***the characteristic equation*** of A.

Example 10.1

Determine the eigenvalues and eigenvectors of the matrix A where $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$. $- \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

Characteristic eq: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(4-\lambda) - (-1) = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 9 = 0$$

$$\Rightarrow (\lambda - 3)^2 = 0$$

$$\Rightarrow \lambda = 3$$

$\therefore$ The only eigen value of A is 3.

$\underline{\lambda = 3}$

$A\vec{x} = 3\vec{x}$

$$\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$$

$$\begin{cases} 2x_1 + x_2 = 3x_1 \\ -x_1 + 4x_2 = 3x_2 \end{cases}$$

$2 - \lambda = 2 - 3$

$$\begin{cases} -1x_1 + 1x_2 = 0 \\ -1x_1 + 1x_2 = 0 \end{cases}$$

$4 - \lambda = 4 - 3$

Clearly $x_2 = x_1$

$\therefore$ An eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Check

$$\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ ✓

Example 10.2
Whence the characteristic equation?

Example 10.3

Determine the eigenvalues for $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -4 & 6 & 0 & 0 \\ -1 & 12 & 3 & 0 \\ 4 & 4 & 2 & 0 \end{bmatrix}$.

10.2 If $\vec{x}$ is an eigenvector of A with the eigenvalue λ, then we have:

$$A\vec{x} = \lambda\vec{x} \implies A\vec{x} = \lambda(I\vec{x})$$
$$\implies A\vec{x} = (\lambda I)\vec{x}$$
$$\implies A\vec{x} - \lambda I\vec{x} = \vec{0}$$
$$\implies (A - \lambda I)\vec{x} = \vec{0} \quad (\text{Eq. } 1)$$

Since $\vec{x}$ cannot be $\vec{0}$, Eq. 1 has non-trivial solutions

$\therefore \quad \det(A - \lambda I) = 0$

Thm. 8, properties d & m.

10.3 // Characteristic eq: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 3-\lambda & 0 & 0 & 0 \\ -4 & 6-\lambda & 0 & 0 \\ -1 & 12 & 3-\lambda & 0 \\ 4 & 4 & 2 & -\lambda \end{vmatrix} = 0 \implies (3-\lambda)(6-\lambda)(3-\lambda)(-\lambda) = 0$$

$\therefore$ our eigenvalues and their algebraic multiplicities are

Table 1: Eigenstuff

eigenvalue	alg. mult.
3	2
6	1
0	1

Definitions 10.4 and 10.5: Eigenspaces (of square matrices)
The set of all eigenvectors associated with the specific eigenvalue λ_i is called the ***λ_i-eigenspace of A***. The dimension of the λ_i-eigenspace is called the geometric multiplicity of λ_i.

Example 10.4

Determine bases for the eigenspaces of B where $B = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$.

Characteristic eq: $\det(B - \lambda I) = 0$

$$\begin{vmatrix} 4-\lambda & -1 & 6 \\ 2 & 1-\lambda & 6 \\ 2 & -1 & 8-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1)^{1+1}(4-\lambda)\begin{vmatrix} 1-\lambda & 6 \\ -1 & 8-\lambda \end{vmatrix} + (-1)^{1+2}(-1)\begin{vmatrix} 2 & 6 \\ 2 & 8-\lambda \end{vmatrix} + (-1)^{1+3}(6)\begin{vmatrix} 2 & 1-\lambda \\ 2 & -1 \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)\left[(1-\lambda)(8-\lambda) - (-6)\right] + \left[2(8-\lambda) - 12\right] + 6\left[-2 - 2(1-\lambda)\right] = 0$$

$$\Rightarrow (4-\lambda)\left[\lambda^2 - 9\lambda + 14\right] + \left[-2\lambda + 4\right] + 6\left[2\lambda - 4\right] = 0$$

$$\Rightarrow (4-\lambda)(\lambda-2)(\lambda-7) - 2(\lambda-2) + 12(\lambda-2) = 0$$

$$\Rightarrow (\lambda-2)\left[(4-\lambda)(\lambda-7) - 2 + 12\right] = 0$$

$$\Rightarrow (\lambda-2)\left[-\lambda^2 + 11\lambda - 18\right] = 0$$

$$\Rightarrow -(\lambda-2)(\lambda^2 - 11\lambda + 18) = 0$$

$$\Rightarrow -(\lambda-2)(\lambda-2)(\lambda-9) = 0$$

$\therefore$ The eigenvalues are 2 (alg mult of 2)
& 9 (alg mult of 1)

2-eigenspace

$$\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1/2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Gen Sol: $\begin{cases} x_1 = \frac{1}{2}x_2 - 3x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$

Basis: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Check

$$\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 2 \end{bmatrix} \checkmark$$

9-eigenspace

$$\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9x_1 \\ 9x_2 \\ 9x_3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -5 & -1 & 6 & 0 \\ 2 & -8 & 6 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Gen Sol: $\begin{cases} x_1 = x_3 \\ x_2 = x_3 \\ x_3 \text{ is free} \end{cases}$

Basis: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Check

$$\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix} \checkmark$$

Definition 10.6 and Theorem 10.1: Similar Matrices

The square matrices A and B are ***similar matrices*** if and only if there exists a matrix P with the property that $A = PBP^{-1}$ (or, similarly, $P^{-1}AP = B$). Similar matrices have the same characteristic equation.

NOTE: Not all matrices that share a characteristic equation are similar!

Theorem 10.2 and Definition 10.: Diagonalization of an $n \times n$ matrix A

If A has n linearly independent eigenvectors, then A is similar to a diagonal matrix, D. Furthermore, $D = P^{-1}AP$ where the columns of P are composed of n linearly independent eigenvectors of A and the main diagonal entry in the i^{th} column of D is the eigenvalue that corresponds to the eigenvector in the i^{th} column of P. The product PDP^{-1} is called a ***diagonalization*** of A.

Example 10.5

Determine two distinct diagonalizations of A where $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$.

The eigenvalues of A are 3 (alg. mult of 2) and 8 (alg. mult of 1). (calculator)

3-eigenspace

$$\begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \\ 3x_3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -1 & -2 & -3 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

General Solution: $\begin{cases} x_1 = -2x_2 - 3x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$

Basis: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Check

$$\begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ 0 \\ 3 \end{bmatrix} \checkmark$$

8 - eigenspace

$$\begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8x_1 \\ 8x_2 \\ 8x_3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -4 & 2 & 3 & 0 \\ -1 & -7 & -3 & 0 \\ 2 & 4 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

General Solution: $\begin{cases} x_1 = \frac{1}{2}x_3 \\ x_2 = -\frac{1}{2}x_3 \\ x_3 \text{ is free} \end{cases}$

Basis: $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$

Check

$$\begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ -8 \\ 16 \end{bmatrix} \checkmark$$

$\therefore$ One diagonalization of A is:

$A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & -2 & -3 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 7 & 3 \\ -2 & -4 & -1 \end{bmatrix}$$

Check: $\begin{bmatrix} 1 & -2 & -3 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 8 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \cdot \frac{1}{5}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 7 & 3 \\ -2 & -4 & -1 \end{bmatrix}$

$$= \frac{1}{5}\begin{bmatrix} 8 & -6 & -9 \\ -8 & 3 & 0 \\ 16 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 7 & 3 \\ -2 & -4 & -1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 20 & 10 & 15 \\ -5 & 5 & -15 \\ 10 & 20 & 45 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix} \checkmark$$

Another diagonalization is

$A = PDP^{-1}$ where:

$$P = \begin{bmatrix} -3 & 1 & -2 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad P^{-1} = \frac{1}{5}\begin{bmatrix} -2 & -4 & -1 \\ 1 & 2 & 3 \\ 1 & 7 & 3 \end{bmatrix}$$

Example10.6

Diagonalize B where $B=\begin{bmatrix}3 & -1 & -2\\ 2 & 0 & -2\\ 2 & -1 & -1\end{bmatrix}$; use the result to simplify B^n where n is a natural number. To get to the point more quickly, let's use our calculators to find the eigenvalues of M.

The eigenvalues of B are 1 (alg mult of 2) and 0 (with an algebraic multiplicity of 1).

1- eigenspace

$$\begin{bmatrix}3 & -1 & -2\\ 2 & 0 & -2\\ 2 & -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}2 & -1 & -2 & 0\\ 2 & -1 & -2 & 0\\ 2 & -1 & -2 & 0\end{array}\right]\sim\left[\begin{array}{ccc|c}1 & -1/2 & -1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

General Solution: $\begin{cases}x_1=\frac{1}{2}x_2+x_3\\ x_2 \text{ is free}\\ x_3 \text{ is free}\end{cases}$

Basis: $\left\{\begin{bmatrix}1\\ 2\\ 0\end{bmatrix},\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}\right\}$

0 - eigenspace

$$\begin{bmatrix}3 & -1 & -2\\ 2 & 0 & -2\\ 2 & -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}3 & -1 & -2 & 0\\ 2 & 0 & -2 & 0\\ 2 & -1 & -1 & 0\end{array}\right]\sim\left[\begin{array}{ccc|c}1 & 0 & -1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

Gen Sol: $\begin{cases}x_1=x_3\\ x_2=x_3\\ x_3 \text{ is free}\end{cases}$ Basis: $\left\{\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}\right\}$

$\therefore \quad B = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix}$$

note: $$B^K = \overbrace{(PDP^{-1})(PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1})}^{K\text{-factors}}$$

$$= PD(P^{-1}P)D(P^{-1}P)D(P^{-1}P)\cdots(P^{-1}P)DP^{-1}$$

$$= PD\,I\,D\,I\,D\,I \cdots I\,D\,P^{-1}$$

$$= P\underbrace{DDD\cdots D}_{K\text{-factors}}P^{-1}$$

$$= PD^KP^{-1}$$

OBServation

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^2 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$= \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$$

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^3 = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$= \begin{bmatrix} a^3 & 0 \\ 0 & b^3 \end{bmatrix}$$

Clearly $$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^K = \begin{bmatrix} a^K & 0 \\ 0 & b^K \end{bmatrix}$$

$$B^k = PD^kP^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1^k & 0 & 0 \\ 0 & 1^k & 0 \\ 0 & 0 & 0^k \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$$

$$= B$$

HOLY CLONES, BATMAN!

$B^K = B$ for all counting numbers, k.

B is called <u>idempotent</u>.

Example 10.7

What happens when we try to diagonalize M where $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$?

Characteristic equation: $\det(M - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{vmatrix} = 0 \Rightarrow 0\,C_{13} + 1\cdot(-1)^{2+3}\begin{vmatrix} -\lambda & 1 \\ 2 & -5 \end{vmatrix} + (4-\lambda)(-1)^{3+3}\begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -[5\lambda - 2] + (4-\lambda)[\lambda^2 - 0] = 0$$

$$\Rightarrow -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0$$

$$\Rightarrow \lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 1)(\lambda - 1) = 0$$

Rational zero Thm (abbreviated)
if the leading coefficient is 1, any rational zero evenly divides into the constant term.
$\{-2, -1, 1, 2\}$

$\therefore$ Table 2: Eigenvalues of M

Value	alg. mult.
1	2
2	1

1 - Eigenspace

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & -5 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

To the chase...

A basis is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Since the 1-eigenspace is the only one that could have geometric multiplicity of 2, and doesn't, M does not have three linearly independent eigenvectors and, hence, cannot be diagonalized.

Example 10.8

Consider the recursive sequence where $a_1 = 1$, $a_2 = 1$, and $a_k = 2a_{k-1} + 3a_{k-2}$ for $k \geq 3$. Let's find a general term formula (non-recursive) for a_k starting at $k = 3$.

$$a_1 = 1, \quad a_2 = 1$$

$$a_3 = 2a_2 + 3a_1 = 2(1) + 3(1) = 5$$

$$a_4 = 2a_3 + 3a_2 = 2(5) + 3(1) = 13$$

note:

$$\begin{bmatrix} a_k \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{k-1} \\ a_{k-2} \end{bmatrix}$$

Define $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$

$$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 3 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 3 = 0$$

$$\Rightarrow \lambda = 3 \text{ or } \lambda = -1$$

$\underline{3\text{-eigenspace}}$

$$\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1 & 3 & 0 \\ 1 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Basis: $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ ✓

$\underline{-1\text{-eigenspace}}$

$$\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & 3 & 0 \\ 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Basis: $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ ✓

$\therefore \quad A = PDP^{-1}$ where $P = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}, P^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$ ✓

$$\therefore \quad A^k = PD^kP^{-1}$$

$$= \frac{1}{4} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & (-1)^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 3^{k+1} & (-1)^{k+1} \\ 3^k & (-1)^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 3^{k+1} + (-1)^{k+2} & 3^{k+1} + 3\cdot(-1)^{k+1} \\ 3^k + (-1)^{k+1} & 3^k + 3\cdot(-1)^k \end{bmatrix}$$

Note

$$\begin{bmatrix} a_3 \\ a_2 \end{bmatrix} = A \begin{bmatrix} a_2 \\ a_1 \end{bmatrix}$$

$$\begin{bmatrix} a_4 \\ a_3 \end{bmatrix} = A \begin{bmatrix} a_3 \\ a_2 \end{bmatrix} = A \cdot A \begin{bmatrix} a_2 \\ a_1 \end{bmatrix} = A^2 \begin{bmatrix} a_2 \\ a_1 \end{bmatrix}$$

$$\begin{bmatrix} a_5 \\ a_4 \end{bmatrix} = A \begin{bmatrix} a_4 \\ a_3 \end{bmatrix} = A \cdot A^2 \begin{bmatrix} a_2 \\ a_1 \end{bmatrix} = A^3 \begin{bmatrix} a_2 \\ a_1 \end{bmatrix}$$

$$\vdots$$

$$\begin{bmatrix} a_k \\ a_{k-1} \end{bmatrix} = A^{k-2} \begin{bmatrix} a_2 \\ a_1 \end{bmatrix}$$

$$\begin{bmatrix} a_k \\ a_{k-1} \end{bmatrix} = A^{k-2} \begin{bmatrix} a_2 \\ a_1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 3^{k-1} + (-1)^k & 3^{k-1} + 3\cdot(-1)^{k-1} \\ 3^{k-2} + (-1)^{k-1} & 3^{k-2} + 3\cdot(-1)^{k-2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \quad a_k = \frac{1}{4}\left(3^{k-1} + (-1)^k + 3^{k-1} + 3(-1)^{k-1}\right)$$

$$= \frac{1}{4}\left(2\cdot 3^{k-1} + (-1)^k - 3(-1)^k\right)$$

$$= \frac{2\cdot 3^{k-1} - 2(-1)^k}{4}$$

Check

$$a_{10} = \frac{2\cdot 3^9 - 2(-1)^{10}}{4}$$

$$= 9,841$$

✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓

9,841

Example 10.9

Diagonalize T where $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Characteristic Eq: $\det(T - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \implies \lambda^2 + 1 = 0$$
$$\implies \lambda^2 = -1$$

Unfortunately, the eigenvalues are i and $-i$ (both alg. multiplicity of one)

i - eigenspace

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} i x_1 \\ i x_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Gen solution: $\begin{cases} x_1 = -i x_2 \\ x_2 \text{ is free} \end{cases}$

Basis: $\left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$

$-i$ - eigenspace

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -i x_1 \\ -i x_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Gen Sol: $\begin{cases} x_1 = i x_2 \\ x_2 \text{ is free} \end{cases}$

Basis: $\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$

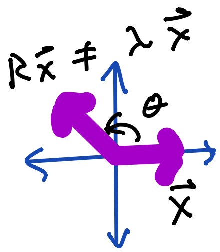
$\therefore$ $T = PDP^{-1}$ where $P = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$,

$P^{-1} = \begin{bmatrix} i/2 & 1/2 \\ -i/2 & 1/2 \end{bmatrix}$

Example 10.10

Explain ***geometrically*** why the rotation matrix $R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ cannot possibly have any real number eigenvalues for $0 < \theta < \pi$.

$$\underbrace{R\vec{x}}_{\text{Changes direction but not length}} = \underbrace{\lambda \vec{x}}_{\text{Changes length but not direction } (\theta \neq \pi)}$$



($\lambda\vec{x}$ points along the x-axis)

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if $\theta = 0$

$\underline{1}$ is an eigenvalue

$R\vec{x} = 1\,\vec{x}$

$\longrightarrow \quad \longrightarrow$

$$R = \begin{bmatrix} \cos(0) & -\sin(0) \\ \sin(0) & \cos(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

if $\theta = \pi$

$\underline{-1}$ is an eigenvalue

$R\vec{x} = -1\,\vec{x}$

$\longrightarrow \quad \longleftarrow$

$$R = \begin{bmatrix} \cos(\pi) & -\sin(\pi) \\ \sin(\pi) & \cos(\pi) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\det(R - \lambda I) = \begin{vmatrix} \lambda+1 & 0 \\ 0 & \lambda+1 \end{vmatrix} = (\lambda+1)^2$$