1. a. The statement is true. $\begin{bmatrix} a \\ b \\ 2a+b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ so $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ obviously forms a basis for $H$. This means any two linearly independent vectors from $H$ form a basis for $H$, and the vectors in $\left\{ \begin{bmatrix} 3 \\ 4 \\ 10 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} \right\}$ meet that criteria.

   b. The statement is false. As shown in part (a), $H$ is a two-dimensional space, so every basis of $H$ contains exactly two vectors. Note: since all three vectors in $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} \right\}$ are in $H$, it must be the case that the set is linearly dependent.

   c. The statement is false. The vector $\begin{bmatrix} 3 \\ 1 \\ 9 \end{bmatrix}$ is not in $H$, so it cannot possibly be in a basis for $H$!

2. a. The statement is self-evidently false. It takes at least five vectors to span $\mathbb{R}^5$ and there are only four columns in A.

   b. The statement is self-evidently true. The column space of $A$ is a subspace of $\mathbb{R}^5$, and any vector in $\mathbb{R}^5$ can be written as a linear combination of $\vec{e}_1 - \vec{e}_5$.

   c. The statement is self-evidently false. The largest dimension that $\operatorname{col}(A)$ can possibly have is four (since $A$ has only four columns), so there is no way a basis for $\operatorname{col}(A)$ can possibly contain five vectors.

   d. The statement is false. This is the first question in the bunch that requires you to find the reduced echelon form of $A$. From that form you see that $A$ has only three pivot columns, so bases for $\operatorname{col}(A)$ contain exactly three vectors.; $\{\vec{C}_1, \vec{C}_2, \vec{C}_3, \vec{C}_4\}$ does not fit that bill.

   e. The statement is true. The vectors in $\{\vec{C}_1, \vec{C}_2, \vec{C}_3\}$ are the pivot columns of $A$.

f. The statement is false. The reduced echelon form of $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 3 & 3 & 0 \\ -2 & 4 & -2 & 0 \\ 5 & 0 & 15 & 0 \\ 0 & 4 & 4 & 0 \end{array}\right]$ shows that $\{\vec{C}_1, \vec{C}_2, \vec{C}_4\}$ is a linearly dependent set, so the three vectors cannot possibly form a basis for anything.

g. The statement is true. The reduced echelon form of $\left[\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 3 & 1 & 3 & 0 \\ 4 & 1 & -2 & 0 \\ 0 & 0 & 15 & 0 \\ 4 & 0 & 4 & 0 \end{array}\right]$ shows that $\{\vec{C}_2, \vec{C}_3, \vec{C}_4\}$ is a linearly independent set. Since the dimension of $\operatorname{col}(A)$ is three, any three linearly independent vectors from $\operatorname{col}(A)$ form a basis for $\operatorname{col}(A)$.

h. The statement is false. The null space of $A$ is a subspace of $\mathbb{R}^4$. (The null space consists of solutions to $A\vec{x} = \vec{0}$; the multiplication doesn't work unless $\vec{x}$ is $4 \times 1$.)

i. The statement is false. Again, the multiplication doesn't work unless $\vec{x}$ is $4 \times 1$, so the domain of $T$ is $\mathbb{R}^4$.

j. The statement is true. The product $A\vec{x}$ is $5 \times 1$, so the images produced by $T$ lie in $\mathbb{R}^5$ which means that the codomain of $T$ is all of $\mathbb{R}^5$.

k. The statement is false. The reduced echelon form of $\left[\begin{array}{cccc|c} 1 & -2 & 2 & 1 & 0 \\ 0 & 3 & 1 & 3 & 0 \\ -2 & 4 & 1 & -2 & 0 \\ 5 & 0 & 0 & 15 & 0 \\ 0 & 4 & 0 & 4 & 1 \end{array}\right]$ shows that there are no solutions to the equation $A\vec{x} = \vec{e}_5$.

1. a. $\begin{bmatrix} 0 & 4 & 3 \\ 1/2 & 0 & 0 \\ 0 & 1/4 & 0 \end{bmatrix}\begin{bmatrix} 18 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 27 \\ 9 \\ 3/2 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} 18 \\ 6 \\ 1 \end{bmatrix}$; ergo, the eigenvalue is $\frac{3}{2}$

b. $\begin{bmatrix} 1 & -2 & 1 \\ -3 & -1 & -2 \\ -7 & 7 & -6 \end{bmatrix}\begin{bmatrix} -5 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -5 \\ 1 \\ 7 \end{bmatrix}$; ergo, the eigenvalue is $0$.

2.

a.

$$\begin{aligned} \det(A-\lambda I)=0 \quad &\Rightarrow \quad \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0 \\ &\Rightarrow \quad (1-\lambda)(3-\lambda)-8=0 \\ &\Rightarrow \quad \lambda^2-4\lambda-5=0 \\ &\Rightarrow \quad (\lambda-5)(\lambda+1)=0 \end{aligned}$$

$\therefore$ The eigenvalues of $A$ are $5$ and $-1$.

b.

$$\begin{aligned} \det(A-\lambda I)=0 \quad &\Rightarrow \quad \begin{vmatrix} -\lambda & 4 \\ -1 & 5-\lambda \end{vmatrix} = 0 \\ &\Rightarrow \quad -\lambda(5-\lambda)+4=0 \\ &\Rightarrow \quad \lambda^2-5\lambda+4=0 \\ &\Rightarrow \quad (\lambda-4)(\lambda-1)=0 \end{aligned}$$

$\therefore$ The eigenvalues of $A$ are $4$ and $1$.

3. a. For $\lambda = 3$:

$$\begin{aligned} (A-\lambda I)\vec{x}=\vec{0} \quad &\Rightarrow \quad \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\Rightarrow \quad x_1 = 2x_2 \end{aligned}$$

$\therefore$ A basis for the $3$-eigenspace is $\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$.

For $\lambda = -2$:

$$\begin{aligned} (A-\lambda I)\vec{x}=\vec{0} \quad &\Rightarrow \quad \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\Rightarrow \quad x_1 = -\frac{1}{2}x_2 \end{aligned}$$

$\therefore$ A basis for the $-2$-eigenspace is $\left\{\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\}$.

b. For $\lambda = 1$:

$$(A - \lambda I)\vec{x} = \vec{0} \quad \Rightarrow \begin{bmatrix} 0 & 0 & 2 \\ -1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 \text{ is free} \\ x_3 = 0 \end{cases}$$

$\therefore$ A basis for the $1$-eigenspace is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda = 3$:

$$(A - \lambda I)\vec{x} = \vec{0} \quad \Rightarrow \begin{bmatrix} -2 & 0 & 2 \\ -1 & -2 & 1 \\ 2 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 2 \\ -1 & -2 & 1 \\ 2 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = 0 \\ x_3 \text{ is free} \end{cases}$$

$\therefore$ A basis for the $3$-eigenspace is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

For $\lambda = -1$:

$$(A - \lambda I)\vec{x} = \vec{0} \quad \Rightarrow \begin{bmatrix} 2 & 0 & 2 \\ -1 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 2 \\ 1 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \begin{cases} x_1 = -x_3 \\ x_2 = -x_3 \\ x_3 \text{ is free} \end{cases}$$

$\therefore$ A basis for the $3$-eigenspace is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

4. $B = PDP^{-1}$ where $P = \begin{bmatrix} 1 & -1 & 2 \\ -3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$, and $P^{-1} = \frac{1}{6}\begin{bmatrix} -1 & -1 & 2 \\ -3 & 3 & 6 \\ 2 & 2 & 2 \end{bmatrix}$

$$\begin{aligned} PDP^{-1} &= \frac{1}{6}\begin{bmatrix} 1 & -1 & 2 \\ -3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} -1 & -1 & 2 \\ -3 & 3 & 6 \\ 2 & 2 & 2 \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} 0 & -6 & 12 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} -1 & -1 & 2 \\ -3 & 3 & 6 \\ 2 & 2 & 2 \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} 42 & 6 & -12 \\ -18 & 18 & 36 \\ 12 & 12 & 12 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 1 & -2 \\ -3 & 3 & 6 \\ 2 & 2 & 2 \end{bmatrix} \quad \checkmark \end{aligned}$$

5. The characteristic equation is $\det(A - \lambda I) = 0$

$$\begin{aligned} \det(A - \lambda I) = 0 \quad &\Rightarrow \quad \begin{vmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0 \\ &\Rightarrow \quad (1-\lambda)(2-\lambda) = 0 \end{aligned}$$

$\therefore$ The eigenvalues of $A$ are $1$ and $2$.

For $\lambda = 1$:

$$\begin{aligned} (A - \lambda I)\vec{x} = \vec{0} \quad &\Rightarrow \quad \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\Rightarrow \quad \begin{cases} x_1 \text{ is free} \\ x_2 = 0 \end{cases} \end{aligned}$$

$\therefore$ A basis for the $1$-eigenspace is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda = 2$:

$$\begin{aligned} (A - \lambda I)\vec{x} = \vec{0} \quad &\Rightarrow \quad \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\Rightarrow \quad x_1 = x_2 \end{aligned}$$

$\therefore$ A basis for the $2$-eigenspace is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Thus $A = PDP^{-1}$ where $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, and $P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ .

So …

$$\begin{aligned} A^k &= PD^k P^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & 2^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2^k \\ 0 & 2^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1+2^k \\ 0 & 2^k \end{bmatrix} \end{aligned}$$

This gives us …

$$\begin{aligned} A^{10} &= \begin{bmatrix} 1 & -1+2^{10} \\ 0 & 2^{10} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1023 \\ 0 & 1024 \end{bmatrix} \end{aligned}$$