MTH 261 Graded HW 7 **Name** Key

This assignment is due at 6:00 PM on Monday, May 22

You may work on this assignment with your classmates or anybody else you please. You may get help from a tutor or even the instructor. What you may not do is simply copy somebody else's work – that completely obviates the purpose of the assignment. If you forget to complete the assignment before it is due, do not simply copy someone else's paper and turn that in … that is not "working together," that is taking credit for somebody else's work. <u>You should not be working on this in class right before it is due; you have a five days to get this done – it should be done well before ten minutes before it is due</u>.

1. Let $\beta = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and $\gamma = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\}$. Determine $\underset{\gamma \leftarrow \beta}{P}$ and use that matrix to find $[\vec{x}]_\gamma$ where $[\vec{x}]_\beta = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$. Show all relevant work, show an explicit check of your result, and make sure that your conclusion is clear. (See example 9.3.)

$$\left[\begin{array}{cc:cc} 1 & 3 & 1 & 1 \\ 3 & 0 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{cc:cc} 1 & 0 & 1 & 1/3 \\ 0 & 1 & 0 & 2/9 \end{array}\right] \therefore \underset{\gamma \leftarrow \beta}{P} = \begin{bmatrix} 1 & 1/3 \\ 0 & 2/9 \end{bmatrix}$$

For $[\vec{x}]_\beta = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$ we have:

$$[\vec{x}]_\gamma = \underset{\gamma \leftarrow \beta}{P} [\vec{x}]_\beta$$

$$= \begin{bmatrix} 1 & 1/3 \\ 0 & 2/9 \end{bmatrix} \begin{bmatrix} -7 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

Check

using $[\vec{x}]_\beta = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$

$$\vec{x} = (-7)\begin{bmatrix} 1 \\ 3 \end{bmatrix} + (9)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -12 \end{bmatrix}$$

using $[\vec{x}]_\gamma = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

$$\vec{x} = (-4)\begin{bmatrix} 1 \\ 3 \end{bmatrix} + (2)\begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -12 \end{bmatrix}$$

2. Find – completely without the use of your calculator – a general formula for M^k where $M = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$. Obviously I will need to see all relevant work. (See example 10.6.)

The characteristic equation is: $\det(M - \lambda I) = 0$

$$\begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)(1-\lambda) + 2 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 3) = 0$$

$$\Rightarrow \lambda = 2 \text{ or } \lambda = 3$$

2-eigenspace

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Gen Sol: $\begin{cases} x_1 = \frac{1}{2}x_2 \\ x_2 \text{ is free} \end{cases}$ Basis: $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

3-eigenspace

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Gen Sol: $\begin{cases} x_1 = x_2 \\ x_2 \text{ is free} \end{cases}$ Basis: $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

$\therefore$ $M = PDP^{-1}$ where $P = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$

$\therefore$ $M^k = PD^kP^{-1}$

$$= \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2^k & 3^k \\ 2^{k+1} & 3^k \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2^k + 2\cdot 3^k & 2^k - 3^k \\ -2^{k+1} + 2\cdot 3^k & 2^{k+1} - 3^k \end{bmatrix}$$

Check

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}^3 = \begin{bmatrix} 46 & -19 \\ 38 & -11 \end{bmatrix}$$

$$\begin{bmatrix} -2^3 + 2\cdot 3^3 & 2^3 - 3^3 \\ -2^4 + 2\cdot 3^3 & 2^4 - 3^3 \end{bmatrix}$$

$$= \begin{bmatrix} 46 & -19 \\ 38 & -11 \end{bmatrix}$$