MTH 261 Graded HW 5 **Name** Key

This assignment is due at 6:00 PM on Wednesday, May 10

<u>You should not be working on this in class right before it is due; you have a week to get this done – it should be done well before ten minutes before it is due</u>.

1. Consider the linear transformations described below.
   - $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ by reflecting vectors across the line $y = x$.
   - $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ by rotating vectors 45° in the clockwise direction.
   - $T_3 : \mathbb{R}^2 \to \mathbb{R}^2$ by reflecting vectors across the line $y = -x$.
   - $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(\vec{x}) = T_3(T_2(T_1(\vec{x})))$.

a. Track the two standard unit vectors from $\mathbb{R}^2$ through the three step T process. Then write down the implied linear transformation matrix, M, where $T(\vec{x}) = M\vec{x}$.

$$\boxed{\begin{bmatrix}1\\0\end{bmatrix}} \xrightarrow{T_1} \boxed{\begin{bmatrix}0\\1\end{bmatrix}} \xrightarrow{T_2} \boxed{\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}} \xrightarrow{T_3} \boxed{\begin{bmatrix}-1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}}$$

$$\boxed{\begin{bmatrix}0\\1\end{bmatrix}} \xrightarrow{T_1} \boxed{\begin{bmatrix}1\\0\end{bmatrix}} \xrightarrow{T_2} \boxed{\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}} \xrightarrow{T_3} \boxed{\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}}$$

$$M = \frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 1\\-1 & -1\end{bmatrix}$$

b. Track the vector $\vec{x} = \begin{bmatrix}-1\\1\end{bmatrix}$ through the three step process T process and then verify that $M\vec{x} = T(\vec{x})$.

$$\boxed{\begin{bmatrix}-1\\1\end{bmatrix}} \xrightarrow{T_1} \boxed{\begin{bmatrix}1\\-1\end{bmatrix}} \xrightarrow{T_2} \boxed{\begin{bmatrix}0\\-\sqrt{2}\end{bmatrix}} \xrightarrow{T_3} \boxed{\begin{bmatrix}\sqrt{2}\\0\end{bmatrix}}$$

$$M\vec{x} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 1\\-1 & -1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}2/\sqrt{2}\\0\end{bmatrix} = \begin{bmatrix}\sqrt{2}\\0\end{bmatrix} \checkmark$$

1. (Continued)

c. Write down the linear transformation matrices, M_1, M_2, and M_3 such that $T_1(\vec{x}) = M_1\vec{x}$, $T_2(\vec{x}) = M_2\vec{x}$, and $T_3(\vec{x}) = M_3\vec{x}$.

$$M_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

d. Multiply the matrices M_1, M_2, and M_3 in the proper order so that the result is the matrix M. Show the work.

$$T(\vec{x}) = T_3(T_2(T_1(\vec{x}))) \Rightarrow M = M_3(M_2 M_1)$$

$$M_3(M_2 M_1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \left(\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$= M$$

2. Suppose that T_1 rotates vectors from $\mathbb{R}^2$ counterclockwise by 120° whereas T_2 rotates vectors from $\mathbb{R}^2$ clockwise by 120°. Furthermore, suppose that T is the linear transformation that applies T_1 followed by T_2 (i.e., $T(\vec{x}) = T_2(T_1(\vec{x}))$).

a. What is the net effect of applying T_1 followed by T_2. That is, what is the relative position of $\vec{x}$ and $T(\vec{x})$?

$T(\vec{x})$ is exactly $\vec{x}$.

b. Given your answer to part (a), what is the linear transformation matrix for T?

$T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (I)

c. Suppose that A_1 and A_2 are the linear transformation matrices with the properties that $T_1(\vec{x}) = A_1\vec{x}$ and $T_2(\vec{x}) = A_2\vec{x}$. Given your answer to (b) and the fact that T can be effected using $T(\vec{x}) = A_2(A_1\vec{x})$, and the fact that $A_2(A_1\vec{x}) = (A_2 A_1)\vec{x}$, what must be the relationship be between A_1 and A_2 ?

$A_2 A_1 = I \Rightarrow A_1$ & A_2 are inverse matrices.

d. Write down the matrices A_1 and A_2 such that $T_1(\vec{x}) = A_1\vec{x}$ and $T_2(\vec{x}) = A_2\vec{x}$; replace the sine and cosine expressions with their actual (exact) values. Then confirm the relationship you stated in part (c).

$$A_1 = \begin{bmatrix} \cos(120^\circ) & -\sin(120^\circ) \\ \sin(120^\circ) & \cos(120^\circ) \end{bmatrix} \qquad A_2 = \begin{bmatrix} \cos(-120^\circ) & -\sin(-120^\circ) \\ \sin(-120^\circ) & \cos(-120^\circ) \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \qquad = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$$

$$A_1 A_2 = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$