MTH 251 - Fall Term 2015
Test 3 – Given November 24, 2015 **Name** Key

1. Several solutions to the equation $x^2 + xy = y^3 + 7$ are shown in Figure 1. Find the equation of the tangent to this curve at the point $(3,2)$. Make sure that you show all of the relevant work that goes into your determination and make sure that your conclusion is clear. (16 points)

$$x^2 + xy = y^3 + 7$$

$$\frac{d}{dx}(x^2 + xy) = \frac{d}{dx}(y^3 + 7)$$

$$2x + 1 \cdot y + x \cdot \frac{dy}{dx} = 3y^2 \frac{dy}{dx}$$

$$2x + y = (3y^2 - x)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x+y}{3y^2 - x}$$

$$\left.\frac{dy}{dx}\right|_{(3,2)} = \frac{8}{9}$$

$\therefore$ The slope of the tangent line at the point $(3,2)$ is $\frac{8}{9}$ and the equation of that line is $y = \frac{8}{9}x - \frac{2}{3}$

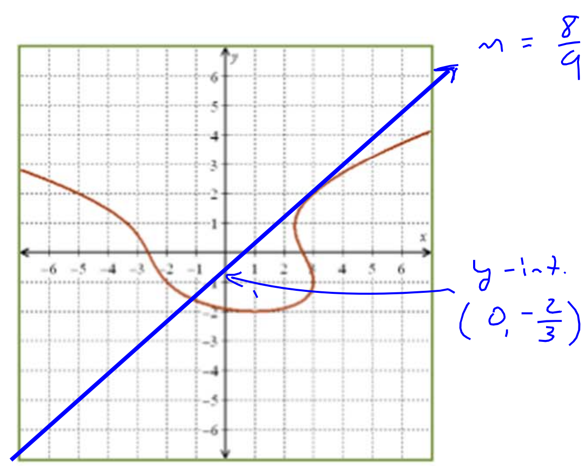


Figure 1: $x^2 + xy = y^3 + 7$

2. Find each indicated derivative making sure that you explicitly show the Leibniz notation step for each application the chain rule. Also, make sure that you do not use the chain rule when the rule is not called for. Make sure that you use proper notation on all steps. This problem continues on page 3. (21 points total)

   a. Find the first derivative with respect to x of the function $f(x)=\sqrt{\sin(\sin(x))}$.

$$f'(x) = \frac{1}{2\sqrt{\sin(\sin(x))}} \cdot \frac{d}{dx}\left(\sin(\sin(x))\right)$$

$$= \frac{1}{2\sqrt{\sin(\sin(x))}} \cdot \cos(\sin(x)) \cdot \frac{d}{dx}(\sin(x))$$

$$= \frac{1}{2\sqrt{\sin(\sin(x))}} \cdot \cos(\sin(x)) \cdot \cos(x)$$

$$= \frac{\cos(\sin(x))\cos(x)}{2\sqrt{\sin(\sin(x))}}$$

   b. Find the first derivative with respect to t of the function $y=\cos(t)\sin^2(t)$

$$\frac{dy}{dt} = \frac{d}{dt}(\cos(t)) \cdot \sin^2(t) + \cos(t) \cdot \frac{d}{dt}(\sin^2(t))$$

$$= -\sin(t)\sin^2(t) + \cos(t) \cdot 2\sin(t) \cdot \frac{d}{dt}(\sin(t))$$

$$= -\sin^3(t) + \cos(t) \cdot 2\sin(t) \cdot \cos(t)$$

$$= 2\sin(t)\cos^2(t) - \sin^3(t)$$

c. $q(x)=\dfrac{x^3-5x}{\ln(x)+1}$

$$q'(x) = \frac{\frac{d}{dx}(x^3-5x)\cdot(\ln(x)+1) - (x^3-5x)\cdot\frac{d}{dx}(\ln(x)+1)}{(\ln(x)+1)^2}$$

$$= \frac{(3x^2-5)(\ln(x)+1) - (x^3-5x)\cdot\frac{1}{x}}{(\ln(x)+1)^2}$$

$$= \frac{3x^2\ln(x)+3x^2-5\ln(x)-5-x^2+5}{(\ln(x)+1)^2}$$

$$= \frac{3x^2\ln(x)-5\ln(x)+2x^2}{(\ln(x)+1)^2}$$

3. Find the first derivative with respect to x of each of the following functions after first completely simplifying the formula being differentiated. No credit will be given if you do not first completely simplify the expression to be differentiated. (8 points)

a. $z=\dfrac{4x^5-3x^3}{2x^3}$

$$z = \frac{4x^5-3x^3}{2x^3}$$

$$= 2x^2-\frac{3}{2}$$

$$\frac{dz}{dx} = 4x$$

b. $g(x)=e^{3\ln(x^2)}$

$$g(x) = e^{3\ln(x^2)}$$

$$= e^{\ln(x^6)}$$

$$= x^6$$

$$g'(x) = 6x^5$$

4. Fill the appropriate derivative formula into each provided blank. ***Make sure that your formula is completely simplified***. **Do not show any work other than the final result.** Do any necessary work on your scratch paper (which I will not be collecting). (21 points total)

a. $\frac{d}{dx}\left(e^{\pi}\right) =$ 0

b. $\frac{d}{dt}\left(\csc(t)\right) =$ $-\csc(t)\cot(t)$

c. $\frac{d}{dx}\left(\ln(3x)\right) =$ $\frac{1}{x}$

d. $\frac{d}{dx}\left(3^{x}\right) =$ $\ln(3)\cdot 3^{x}$

e. $\frac{d}{dx}\left(e^{x/3}\right) =$ $\frac{1}{3}e^{x/3}$

f. $\frac{d}{dt}\left(\sin^{8}(t)\right) =$ $8\sin^{7}(t)\cos(t)$

g. $\frac{d}{d\theta}\left(\theta\sin(\theta)\right) =$ $\sin(\theta)+\theta\cos(\theta)$

h. $\frac{d}{dx}\left(\frac{8}{\sqrt[8]{x^{7}}}\right) =$ $-\frac{7}{8\sqrt[8]{x^{15}}}$

i. $\frac{d}{dx}\left(\tan^{-1}\left(x^{2}\right)\right) =$ $\frac{2x}{1+x^{4}}$

j. $\frac{d}{dx}\left(\frac{5x^{13}-x^{7}}{x^{7}}\right) =$ $30x^{5}$

k. $\frac{d}{dx}\left(\ln\left(e^{2}\right)\right) =$ 0

l. $\frac{d}{d\beta}\left(\frac{1}{\cos(\beta)}\right) =$ $\sec(\beta)\tan(\beta)$

5. Several function values are given in Table 1. Find the value of $h'(2)$ if $h(x)=x^{2}g\left(x^{2}\right)$. Make sure that you show work that fully supports your answer. (8 points)

Table 1: Function Values

x	$g(x)$	$g'(x)$
2	2	3
3	8	0.5
4	2	-5

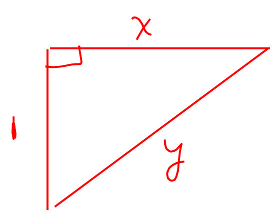
$$h'(x) = \frac{d}{dx}(x^2)g(x^2) + x^2\frac{d}{dx}(g(x^2))$$
$$= 2xg(x^2) + x^2g'(x^2)\cdot\frac{d}{dx}(x^2)$$
$$= 2xg(x^2) + x^2g'(x^2)\cdot 2x$$
$$= 2xg(x^2) + 2\cdot x^3g'(x^2)$$
$$h'(2) = 4g(4) + 16g'(4)$$
$$= 4(2) + 16(-5)$$
$$= -72$$

6. A plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/hr passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when the distance between the plane and the station is 2 mi. (16 points)

Note: To earn full credit for this problem you need to show work consistent with that illustrated and discussed in lecture and lab. This includes, but is not limited to, clear and explicit definitions of the variables in the problem.

Table 1: Variable definitions

name	description	unit
t	Amount of time that has elapsed since the plane flew over the tower	hr
x	length indicated in the diagram at t	mi
y	length indicated in the diagram at t	mi
dx/dt	rate at which x changes with respect to time	mi/hr
dy/dt	rate at which y changes with respect to time	mi/hr



$$1^2 + x^2 = y^2$$

$$x^2 + 1 = y^2$$

$$\frac{d}{dt}(x^2+1) = \frac{d}{dt}(y^2)$$

$$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$$

Table 2: Knowns and unknowns when the plane is 2 miles from the station

Var.	known/unknown
x	$\sqrt{3}$
y	2
dx/dt	500
dy/dt	unknown

$$\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$$

$$\left.\frac{dy}{dt}\right|_{y=2,\ \frac{dx}{dt}=500} = \frac{\sqrt{3}}{2}(500)$$

$$= 250\sqrt{3}$$

$\therefore$ when the plane is 2 mi from the radar station that distance is increasing at the rate of $250\sqrt{3}$ mph.

7. Use the process of logarithmic differentiation to find a formula for $\frac{dy}{dx}$ where $y = x^{x^2 \ln(x^5)}$. To earn full credit you need to fully expand all logarithmic expressions before taking the derivative. (10 points)

$$y = x^{x^2 \ln(x^5)}$$

$$\ln(y) = \ln\left(x^{x^2 \ln(x^5)}\right)$$

$$\ln(y) = x^2 \ln(x^5) \cdot \ln(x)$$

$$\ln(y) = 5x^2(\ln(x))^2$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}\left(5x^2(\ln(x))^2\right)$$

$$\frac{1}{y}\frac{dy}{dx} = 10x(\ln(x))^2 + 5x^2 \cdot 2\ln(x) \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = y\left[10x(\ln(x))^2 + 10x\ln(x)\right]$$

$$= x^{x \ln(x^5)}\left[10x(\ln(x))^2 + 10x\ln(x)\right]$$

$$= 10x\ln(x)(\ln(x)+1) \cdot x^{x\ln(x^5)}$$