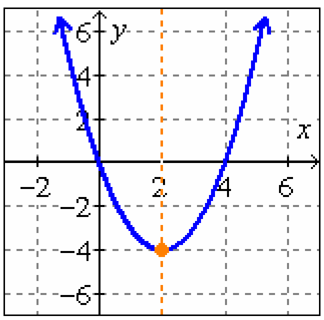
Section V: Quadratic Equations and Functions

Module 3: Graphing Quadratic Functions

In this module, we'll review the graphing quadratic functions (you should have studied the graphs of quadratic functions in your Introductory Algebra courses) and we'll discuss how we can use the completing-the-square technique to help us graph quadratic functions. As we know from our previous course-work, the graph of every quadratic function is a **parabola** and a parabola contains a **vertex**. (The vertex of a parabola is the point where the curve reaches the maximum or minimum output value.) Further, we know that a parabola is symmetric about a vertical line that passes through its vertex; this line is called the **axis of symmetry** for the parabola. In Example 1 we'll consider the graphs of a few quadratic functions.

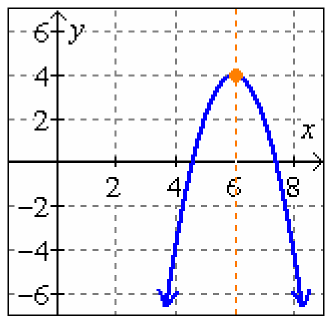
EXAMPLE 1:

a. $a(x) = (x-2)^2 - 4$



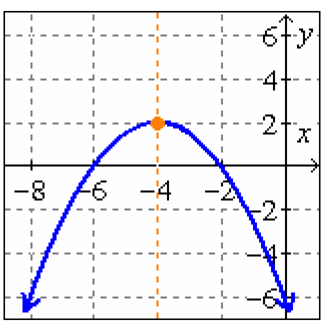
Vertex: $(2, -4)$
Axis of Symmetry: $x = 2$

b. $b(x) = -2(x-6)^2 + 4$



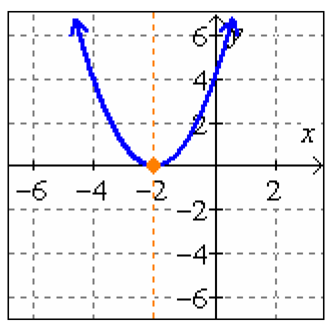
Vertex: $(6, 4)$
Axis of Symmetry: $x = 6$

c. $c(x) = -\frac{1}{2}(x+4)^2 + 2$



Vertex: $(-4, 2)$
Axis of Symmetry: $x = -4$

d. $d(x) = (x+2)^2$



Vertex: $(-2, 0)$
Axis of Symmetry: $x = -2$

Notice that the algebraic rules for all four of the functions in Example 1 have the form "$a(x-h)^2+k$" and notice that the vertex of each of the four graphs is the point (h, k) and that the axis of symmetry for each graph is the line $x = h$. Further notice that if $a > 0$ then the parabola opens up and if $a < 0$ then the parabola opens down. Let's summarize this information:

> The graph of the quadratic function $f(x) = a(x-h)^2 + k$ is a **parabola** with **vertex** (h, k) and **axis of symmetry** the line $x = h$. If $a > 0$ then the parabola **opens up** and if $a < 0$ then the parabola **opens down**.

EXAMPLE 2: Graph the function $g(x) = (x+5)^2 + 10$.

SOLUTION:

Since $g(x) = \left(x-(-5)\right)^2 + 10$, the vertex of $y = g(x)$ is $(h, k) = (-5, 10)$ and the axis of symmetry is the line $x = -5$. To draw a good graph, we should find a few of additional points; the easiest point to find is the y-intercept. Since y-intercept occurs when $x = 0$, we simply need to find $g(0)$:

$$\begin{aligned} g(0) &= (0+5)^2 + 10 \\ &= 35 \end{aligned}$$

So $(0, 35)$ is the y-intercept of $y = g(x)$. Now we can use the axis of symmetry to find another point on the graph of $y = g(x)$. Since $(0, 35)$ occurs 5 units to the *right* of the axis of symmetry, there must be another point on the graph that is 5 units to the *left* of the axis of symmetry with has the same y–value, 35. Thus, the point $(-10, 35)$ must be on the graph. Let's plot all of this information, and then connect our points to complete the graph of $y = g(x)$:

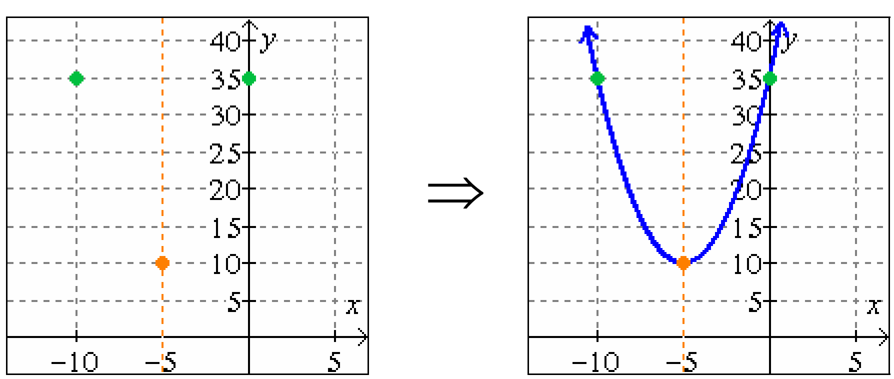


Figure 1: Graphing $g(x) = (x+5)^2 + 10$.

EXAMPLE 3: Graph the function $m(x) = 2(x - 3)^2 - 6$.

SOLUTION:

Since $m(x) = 2(x - 3)^2 + (-6)$ the vertex of $y = m(x)$ is $(h, k) = (3, -6)$ and the axis of symmetry is $x = 3$. To draw a good graph, we should find a few of additional points; the easiest point to find is the y-intercept. Since y-intercept occurs when $x = 0$, we simply need to find $m(0)$:

$$\begin{aligned} m(0) &= 2(0 - 3)^2 - 6 \\ &= 12 \end{aligned}$$

So $(0, 12)$ is the y-intercept of $y = m(x)$. Now we can use the axis of symmetry to find another point on the graph of $y = m(x)$. Since $(0, 12)$ occurs 3 units to the *left* of the axis of symmetry, there must be another point on the graph that is 3 units to the *right* of the axis of symmetry with has the same y–value, 12. Thus, the point $(6, 12)$ must be on the graph. Let's plot all of this information, and then connect our points to complete the graph of $y = m(x)$:

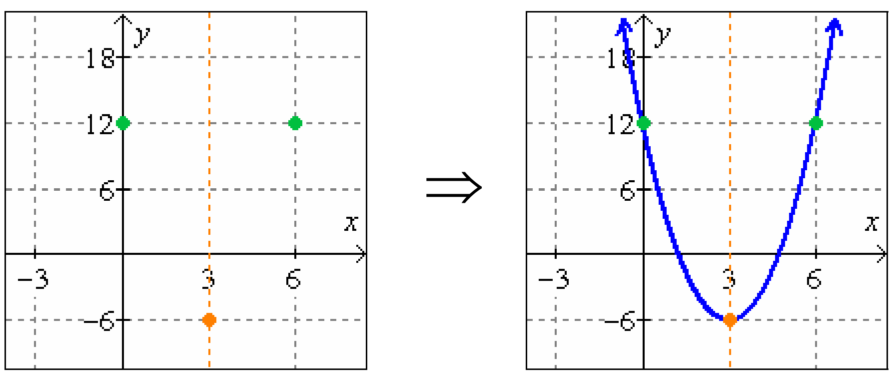


Figure 2: Graphing $m(x) = 2(x - 3)^2 - 6$.

EXAMPLE 4: Find an algebraic rule for a quadratic function q that has vertex $(-3, 5)$ and y-intercept $(0, 41)$.

SOLUTION:

Since the vertex is $(-3, 5)$, we know that the function has form

$$\begin{aligned} q(x) &= a(x - h)^2 + k \\ &= a\left(x - (-3)\right)^2 + 5 \\ &= a\left(x + 3\right)^2 + 5. \end{aligned}$$

We can find a by using the y-intercept $(0,\ 41)$:

$$
\begin{aligned}
(0,\ 41) \ \Rightarrow\ q(0) &= 41 = a\left(0+3\right)^2 + 5 \\
\Rightarrow\qquad 41 &= 9a + 5 \\
\Rightarrow\qquad 36 &= 9a \\
\Rightarrow\qquad a &= \tfrac{36}{9} = 4
\end{aligned}
$$

Thus, $q(x) = 4\left(x+3\right)^2 + 5$.

EXAMPLE 5: Graph the function $p(x) = x^2 - 8x + 14$.

SOLUTION:

The algebraic rule for this function doesn't have form "$a(x-h)^2 + k$" so we can't use what we've studied to find the vertex and axis of symmetry but our experience tells us that the graph of $p(x) = x^2 - 8x + 14$ is parabola so it must have a vertex and an axis of symmetry, so we should be able find them and write the function in the form $a(x-h)^2 + k$.

The function $p(x) = x^2 - 8x + 14$ is said to have form "$ax^2 + bx + c$," and it turns out that we can use the completing-the-square technique to convert a function in the form "$ax^2 + bx + c$" in to a function with the form "$a(x-h)^2 + k$". (To review the completing-the-square process, study Section V: Module 1.)

$$
\begin{aligned}
p(x) &= x^2 - 8x + 14 \\
&= (x^2 - 8x + 16 - 16) + 14 && \text{(we add and subtract } \left(\tfrac{-8}{2}\right)^2 = 16\text{)} \\
&= (x^2 - 8x + 16) - 16 + 14 && \text{(we group terms so that we can complete the square)} \\
&= (x-4)^2 - 2
\end{aligned}
$$

Thus, the vertex of $y = p(x)$ is $(4,\ -2)$ and the axis of symmetry is $x = 4$. To draw a good graph, let's find the y-intercept and then use the axis of symmetry to find an additional point:

$$
\begin{aligned}
p(0) &= 0^2 - 8(0) + 14 \\
&= 14
\end{aligned}
$$

So the y-intercept is $(0, 14)$, and since this point is 4 units to the left of the axis of symmetry, we know we need a corresponding point 4 units to the right of the axis of symmetry, so the point $(8, 14)$ must also be on the curve. Let's plot all of this information, and then connect our points to complete the graph of $y = p(x)$:

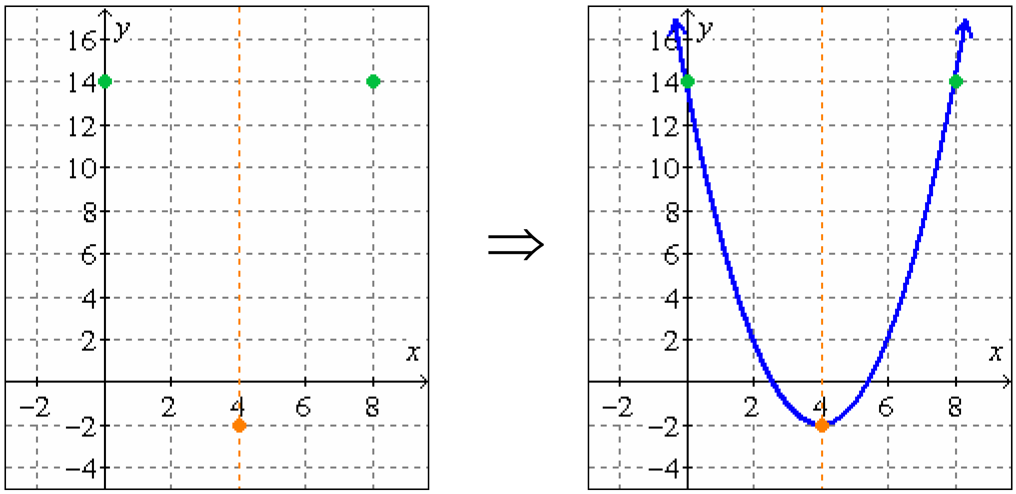


Figure 3: Graphing $p(x) = x^2 - 8x + 14$.

EXAMPLE 6: Graph the function $w(x) = -2x^2 - 40x - 160$.

SOLUTION:

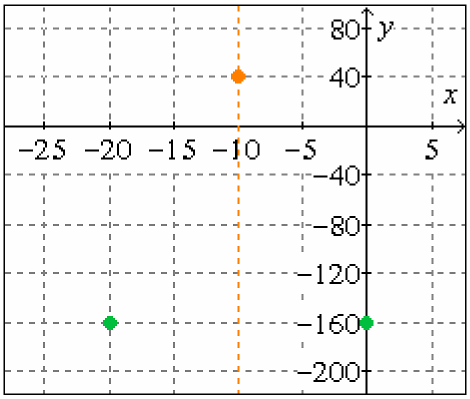
Let's complete the square in order to convert the function to the form "$a(x - h)^2 + k$" so we can determine its vertex and axis of symmetry.

$$\begin{aligned} w(x) &= -2x^2 - 40x - 160 \\ &= -2(x^2 + 20x) - 160 \\ &= -2\left(x^2 + 20x + 100 - 100\right) - 160 && \text{(we add and subtract } \left(\tfrac{20}{2}\right)^2 = 100\text{)} \\ &= -2\left(x^2 + 20x + 100\right) + (-2)(-100) - 160 && \text{(we group terms so that we can complete the square)} \\ &= -2\left(x + 10\right)^2 + 40 \\ &= -2\left(x - (-10)\right)^2 + 40 \end{aligned}$$

Thus, the vertex of $y = w(x)$ is $(-10, 40)$ and the axis of symmetry is $x = -10$. To draw a good graph, let's find the y-intercept and then use the axis of symmetry to find an additional point:

$$\begin{aligned} w(0) &= -2(0)^2 - 40(0) - 160 \\ &= -160 \end{aligned}$$

So the y-intercept is $(0, -160)$, and since this point is 10 units to the right of the axis of symmetry, we know we need a corresponding point 10 units to the left of the axis of symmetry, so the point $(20, -160)$ must also be on the curve. Let's plot all of this information, and then connect our points to complete the graph of $y = w(x)$:



$\Rightarrow$

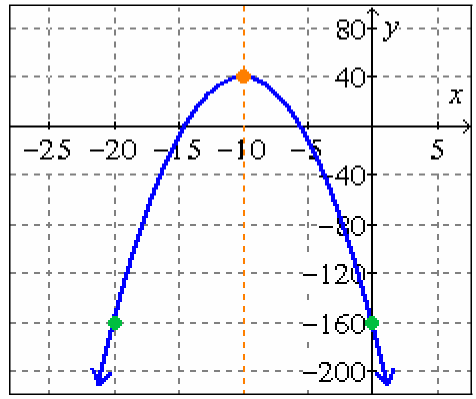


Figure 4: Graphing $w(x) = -2x^2 - 40x - 160$.

EXAMPLE 7: Use the completing-the-square technique to find the vertex of the generic quadratic function $f(x) = ax^2 + bx + c$.

SOLUTION:

$$
\begin{aligned}
f(x) &= ax^2 + bx + c \\
&= a\left(x^2 + \frac{b}{a}x\right) + c \\
&= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c \\
&= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) - a\left(\frac{b}{2a}\right)^2 + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b^2}{4a^2}\right) + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \cdot \frac{4a}{4a} \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + \frac{4ac}{4a} \\
&= a\left(x - \left(-\frac{b}{2a}\right)\right)^2 + \frac{4ac - b^2}{4a}
\end{aligned}
$$

Thus, the vertex of $f(x) = ax^2 + bx + c$ is the point $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$.

Note that since the y-coordinate of vertex is the output for the function when the input is the x-coordinate of the vertex. Thus, the y-coordinate can be found by calculating $f\left(-\frac{b}{2a}\right)$. Let's summarize this information:

The graph of the quadratic function $f(x) = ax^2 + bx + c$ is a **parabola** with **vertex** $\left(-\frac{b}{2a},\ f\left(-\frac{b}{2a}\right)\right)$ and **axis of symmetry** the line $x = -\frac{b}{2a}$.

EXAMPLE 8: Graph the function $j(x) = 3x^2 + 12x + 8$.

SOLUTION:

First notice that $j(x) = 3x^2 + 12x + 8$ has the form "$ax^2 + bx + c$" where $a = 3$, $b = 12$, and $c = 8$. Let's find the x-coordinate of the vertex, $-\frac{b}{2a}$:

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{12}{2(3)} \\ &= -2 \end{aligned}$$

Now let's find the y-coordinate of the vertex, $j(-2)$:

$$\begin{aligned} j(-2) &= 3(-2)^2 + 12(-2) + 8 \\ &= 12 - 24 + 8 \\ &= -4 \end{aligned}$$

Thus, the vertex is $(-2, -4)$. We can find the y-intercept by calculating $j(0)$:

$$\begin{aligned} j(0) &= 3(0)^2 + 12(0) + 8 \\ &= 8 \end{aligned}$$

So the y-intercept is $(0, 8)$. Since the y-intercept is 2 units to the right of the axis of symmetry, we know that there must be a corresponding point 2 units to the left of the axis of symmetry, so the point $(-4, 8)$ must also be on the graph of $y = j(x)$. Let's plot all of this information, and then connect our points to complete the graph of $y = j(x)$:

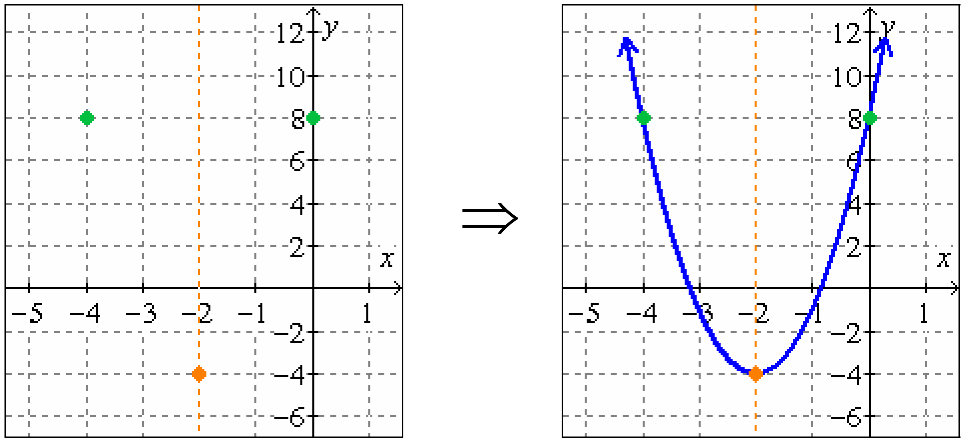


Figure 5: Graphing $j(x) = 3x^2 + 12x + 8$.