

## Section V: Quadratic Equations and Functions

### Module 1: Solving Quadratic Equations Using *Factoring*, *Square Roots*, *Graphs*, and *Completing-the-Square*



**DEFINITION:** A quadratic equation is an equation of the form  $ax^2 + bx + c = 0$  where  $a$ ,  $b$ , and  $c$  are real numbers and  $a \neq 0$ .

In your introductory algebra course, you should have solved quadratic equations using factoring, graphs, the quadratic formula, and the square root method. In this module, we will review solving quadratic equations using factoring, using graphs, and using the square root method, as well as introduce solving quadratic equations by completing the square. In the next module we will study solving quadratic equations using the quadratic formula.

### SOLVING QUADRATIC EQUATIONS BY *FACTORING*



**EXAMPLE:** Solve the following quadratic equations by *factoring*.

a.  $x^2 - 5x + 6 = 0$

b.  $t^2 = 16$

c.  $5z^2 = 2 - 9z$

**SOLUTIONS:**

a.  $x^2 - 5x + 6 = 0$   
 $\Rightarrow (x - 3)(x - 2) = 0$   
 $\Rightarrow x - 3 = 0 \quad \text{or} \quad x - 2 = 0$   
 $\Rightarrow x = 3 \quad \text{or} \quad x = 2$

Thus, the solution set is  $\{3, 2\}$

**b.**  $t^2 = 16$

$$\Rightarrow t^2 - 16 = 0$$

$$\Rightarrow (t + 4)(t - 4) = 0$$

$$\Rightarrow t + 4 = 0 \quad \text{or} \quad t - 4 = 0$$

$$\Rightarrow t = -4 \quad \text{or} \quad t = 4$$

Thus, the solution set is  $\{-4, 4\}$

**c.**  $5z^2 = 2 - 9z$

$$\Rightarrow 5z^2 + 9z - 2 = 0$$

$$\Rightarrow (5z - 1)(z + 2) = 0$$

$$\Rightarrow 5z - 1 = 0 \quad \text{or} \quad z + 2 = 0$$

$$\Rightarrow 5z = 1 \quad \text{or} \quad z = -2$$

$$\Rightarrow z = \frac{1}{5} \quad \text{or} \quad z = -2$$

Thus, the solution set is  $\left\{\frac{1}{5}, -2\right\}$

## SOLVING QUADRATIC EQUATIONS USING THE SQUARE ROOT METHOD



**EXAMPLE:** Solve the following quadratic equations using *the square root method*.

**a.**  $x^2 = 16$

**c.**  $(y - 7)^2 = 9$

**b.**  $p^2 = -25$

**d.**  $(z + 3)^2 + 7 = 0$

**SOLUTIONS:**

**a.**  $x^2 = 16$

$$\Rightarrow \sqrt{x^2} = \pm \sqrt{16}$$

$$\Rightarrow x = \pm 4$$

Thus, the solution set is  $\{-4, 4\}$ .

**b.**  $p^2 = -25$   
 $\Rightarrow \sqrt{p^2} = \pm \sqrt{-25}$   
 $\Rightarrow p = \pm 5i$

Thus, the solution set is  $\{5i, -5i\}$ .

**c.**  $(y - 7)^2 = 9$   
 $\Rightarrow \sqrt{(y - 7)^2} = \pm \sqrt{9}$   
 $\Rightarrow y - 7 = \pm 3$   
 $\Rightarrow y = 7 \pm 3$   
 $\Rightarrow y = 10 \text{ or } y = 4$

Thus, the solution set is  $\{10, 4\}$ .

**d.**  $(z + 3)^2 + 7 = 0$   
 $\Rightarrow (z + 3)^2 = -7$   
 $\Rightarrow \sqrt{(z + 3)^2} = \pm \sqrt{-7}$   
 $\Rightarrow z + 3 = \pm i\sqrt{7}$   
 $\Rightarrow z = -3 \pm i\sqrt{7}$

Thus, the solution set is  $\{-3 + i\sqrt{7}, -3 - i\sqrt{7}\}$ .



**EXAMPLE:** Let  $f(x) = x + 5$  and  $g(x) = \frac{3}{x - 5}$ . Find all  $x$  for which  $f(x) = g(x)$ .

**SOLUTION:**

$$f(x) = g(x)$$

$$\Rightarrow x + 5 = \frac{3}{x - 5}$$

This isn't a quadratic equation. In fact, it is a *rational equation*. But in order to solve such an equation, we need to clear the fractions and, once we clear the fractions, we have a quadratic equation:

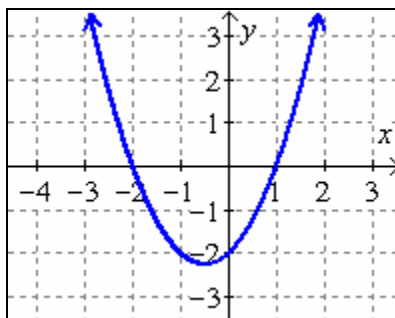
$$\begin{aligned}
 x + 5 &= \frac{3}{x - 5} \\
 \Rightarrow (x + 5) \cdot (x - 5) &= \frac{3}{x - 5} \cdot (x - 5) && \text{(to clear the fraction, multiply both sides of the equation by } x - 5) \\
 \Rightarrow x^2 - 25 &= 3 \\
 \Rightarrow x^2 &= 28 \\
 \Rightarrow \sqrt{x^2} &= \pm \sqrt{28} && \text{(since there isn't a linear term, we use the square root method)} \\
 \Rightarrow x &= \pm \sqrt{4 \cdot 7} && \text{(factor 28)} \\
 \Rightarrow x &= \pm 2\sqrt{7}
 \end{aligned}$$

Thus, the solution set is  $\{2\sqrt{7}, -2\sqrt{7}\}$ .

## SOLVING QUADRATIC EQUATIONS USING GRAPHS



**EXAMPLE:** Use the graph below to solve the quadratic equation  $x^2 + x - 2 = 0$ .



The graph of  $y = x^2 + x - 2$ .

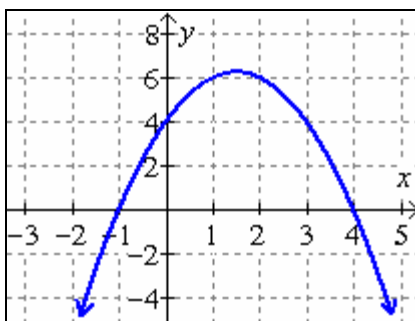
**SOLUTION:**

Since we are given the graph of  $y = x^2 + x - 2$  and we are asked to solve  $x^2 + x - 2 = 0$ , we need to look for any places on the graph where  $y = 0$ , i.e., we need to find the  $x$ -intercepts of the graph. Since the  $x$ -intercepts are  $(-2, 0)$  and  $(1, 0)$ , we can conclude that the solutions are  $x = -2$  or  $x = 1$ , so the solution set is  $\{-2, 1\}$ .



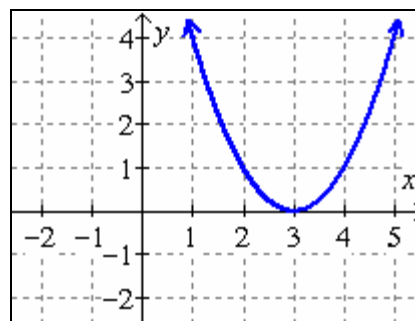
**EXAMPLE:** Use the graphs below to solve the given quadratic equations.

a.  $0 = -x^2 + 3x + 4$



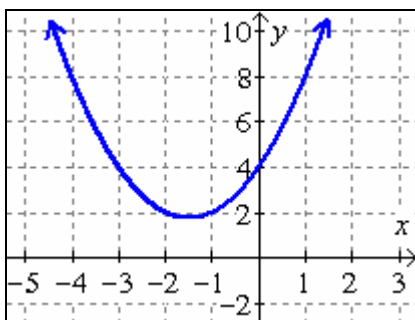
The graph of  $y = -x^2 + 3x + 4$ .

b.  $x^2 - 6x + 9 = 0$



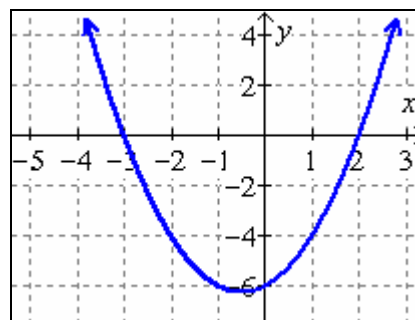
The graph of  $y = x^2 - 6x + 9$ .

c.  $x^2 + 3x + 4 = 0$



The graph of  $y = x^2 + 3x + 4$ .

d.  $x^2 = 6 - x$



The graph of  $y = x^2 + x - 6$ .

### SOLUTIONS:

- a. Since we are given the graph of  $y = -x^2 + 3x + 4$  and we are asked to solve  $0 = -x^2 + 3x + 4$ , we need to look for any places where the graph of  $y = -x^2 + 3x + 4$  intersects the line  $y = 0$ , i.e., we need to find the  $x$ -intercepts of the graph. Since the  $x$ -intercepts are  $(-1, 0)$  and  $(4, 0)$ , we can conclude that the solutions are  $x = -1$  or  $x = 4$ , so the solution set is  $\{-1, 4\}$ .
- b. Since the graph of  $y = x^2 - 6x + 9$  intersects the line  $y = 0$  only when  $x = 3$ , we can conclude that the solution set is  $\{3\}$ .
- c. Since the graph of  $y = x^2 + 3x + 4$  never intersects the line  $y = 0$  (i.e., the  $x$ -axis), we see that " $x^2 + 3x + 4$ " never equals 0, so we can conclude that the equation  $x^2 + 3x + 4 = 0$  does not have any real solutions. (Note that we can use the quadratic formula or completing-the-square to find the complex numbers that solve the equation.)

- d. Here, we are given the graph of  $y = x^2 + x - 6$  so we need to first obtain the expression " $x^2 + x - 6$ " in the given equation  $x^2 = 6 - x$  which we can do by solving this equation for 0:

$$x^2 = 6 - x$$

$$\Rightarrow x^2 + x - 6 = 0 \quad (\text{subtract } 6 - x \text{ from both sides})$$

We can solve this equation by looking for the  $x$ -values where the graph of  $y = x^2 + x - 6$  intersects the line  $y = 0$  (i.e., the  $x$ -axis). Thus, the solution set is  $\{-3, 2\}$

## SOLVING QUADRATIC EQUATIONS BY COMPLETING-THE-SQUARE

Using the square root method to solve a quadratic equation only works if we can write the quadratic equation so that one side is the square of a binomial. To write the square of a binomial, we start with a perfect square trinomial, i.e., a trinomial that can be factored into two identical binomial factors).

| <i>Perfect Square Trinomials</i> | $\Rightarrow$ | <i>Factored Form</i> |
|----------------------------------|---------------|----------------------|
| $x^2 + 8x + 16$                  | $\Rightarrow$ | $(x + 4)^2$          |
| $x^2 - 6x + 9$                   | $\Rightarrow$ | $(x - 3)^2$          |

Notice that for each perfect square trinomial, the constant term of the trinomial is the square of half the linear term (in this case, the coefficient of  $x$ ). For example,

$$x^2 + 8x + 16$$

$$\frac{1}{2} \cdot (8) = 4 \text{ and } 4^2 = 16$$

$$x^2 - 6x + 9$$

$$\frac{1}{2}(-6) = -3 \text{ and } (-3)^2 = 9$$

The process of writing a quadratic equation so that one side is a perfect square trinomial is called **completing the square**.



**EXAMPLE:** Solve  $x^2 + 2x - 3 = 0$  for  $x$  by completing the square.

**SOLUTION:**

First, we need to find the square of half the coefficient of the linear term (in this case, the coefficient of  $x$ ):

$$\frac{1}{2} \cdot (2) = 1 \text{ and } 1^2 = 1$$

Now we add and subtract 1 on the left-hand side of the equation. Notice that we are just adding zero in a disguised form, so we won't change the solution to the quadratic equation.

$$\begin{aligned} x^2 + 2x + 1 - 1 - 3 &= 0 && \text{(add and subtract 1 on the left-hand side)} \\ \Rightarrow (x^2 + 2x + 1) - 1 - 3 &= 0 \\ \Rightarrow (x + 1)(x + 1) - 4 &= 0 \\ \Rightarrow (x + 1)^2 - 4 &= 0 \\ \Rightarrow (x + 1)^2 &= 4 \\ \Rightarrow \sqrt{(x + 1)^2} &= \pm \sqrt{4} \\ \Rightarrow x + 1 &= \pm 2 \\ \Rightarrow x &= -1 \pm 2 \\ \Rightarrow x = 1 &\text{ or } x = -3 \end{aligned}$$

Thus, the solution set is  $\{1, 3\}$ .



**EXAMPLE:** Solve  $p^2 - 6p + 4 = 0$  for  $p$  by completing-the-square.

**SOLUTION:**

First, we need to find the square of half the coefficient of the linear term (in this case, the coefficient of the term  $-6p$ ).

$$\begin{aligned} \left(\frac{1}{2} \cdot (-6)\right)^2 &= (-3)^2 \\ &= 9 \end{aligned}$$

Now we add and subtract 9 on the left-hand side of the equation.

$$\begin{aligned}
& p^2 - 6p + 4 = 0 \\
\Rightarrow & p^2 - 6p + 9 - 9 + 4 = 0 \\
\Rightarrow & (p^2 - 6p + 9) - 9 + 4 = 0 \\
\Rightarrow & (p - 3)(p - 3) - 5 = 0 \\
\Rightarrow & (p - 3)^2 - 5 = 0 \\
\Rightarrow & (p - 3)^2 = 5 \\
\Rightarrow & \sqrt{(p - 3)^2} = \pm\sqrt{5} \\
\Rightarrow & p - 3 = \pm\sqrt{5} \\
\Rightarrow & p = 3 \pm \sqrt{5}
\end{aligned}$$

Thus, the solution set is  $\{3 + \sqrt{5}, 3 - \sqrt{5}\}$ .

### To Solve a Quadratic Equation by Completing-the-Square:

1. Write the equation in standard form ( $ax^2 + bx + c = 0$ ).
2. Divide both sides of the equation by the coefficient of the quadratic term,  $a$ , in order to make that coefficient 1.
3. Complete the square:
  - a. Multiply the coefficient of the linear term by one-half, then square the result.
  - b. Add and subtract the value obtained in step a.
4. Write the square of a binomial plus a constant, then move the constant to the other side of the equation.
5. Use the square root method to find the solution.



**EXAMPLE:** Solve  $2x^2 - x - 3 = 0$  by completing-the-square.

**SOLUTION:** First, we must divide both sides of the equation by 2 (the coefficient of  $x^2$ ) in order to make 1 the coefficient of  $x^2$ .



$$\begin{aligned}\frac{2x^2 - x - 3}{2} &= \frac{0}{2} \\ \Rightarrow x^2 - \frac{x}{2} - \frac{3}{2} &= 0 \\ \Rightarrow x^2 - \frac{1}{2}x - \frac{3}{2} &= 0\end{aligned}$$

Next, we find the square of half the coefficient of the linear term (in this case, the coefficient of  $x$ ):

$$\begin{aligned}\left(\frac{1}{2} \cdot \left(-\frac{1}{2}\right)\right)^2 &= \left(-\frac{1}{4}\right)^2 \\ &= \frac{1}{16}\end{aligned}$$

Now add and subtract  $\frac{1}{16}$  from the left-hand side and solve for  $x$ :

$$\begin{aligned}x^2 - \frac{x}{2} - \frac{3}{2} &= 0 \\ \Rightarrow x^2 - \frac{x}{2} + \frac{1}{16} - \frac{1}{16} - \frac{3}{2} &= 0 \\ \Rightarrow \left(x^2 - \frac{x}{2} + \frac{1}{16}\right) - \frac{1}{16} - \frac{3}{2} \cdot \frac{8}{8} &= 0 \\ \Rightarrow \left(x - \frac{1}{4}\right)\left(x - \frac{1}{4}\right) - \frac{1}{16} - \frac{24}{16} &= 0 \\ \Rightarrow \left(x - \frac{1}{4}\right)^2 - \frac{25}{16} &= 0 \\ \Rightarrow \left(x - \frac{1}{4}\right)^2 &= \frac{25}{16} \\ \Rightarrow \sqrt{\left(x - \frac{1}{4}\right)^2} &= \pm \sqrt{\frac{25}{16}} \\ \Rightarrow x - \frac{1}{4} &= \pm \frac{5}{4} \\ \Rightarrow x &= \frac{1}{4} \pm \frac{5}{4} \\ \Rightarrow x &= \frac{6}{4} \quad \text{or} \quad x = -\frac{4}{4} \\ \Rightarrow x &= \frac{3}{2} \quad \text{or} \quad x = -1\end{aligned}$$

Thus, the solution set is  $\left\{\frac{3}{2}, -1\right\}$ .



**EXAMPLE:** Solve  $x^2 + 3x + 4 = 0$  by completing the square. [Recall that we tried to solve this equation graphically and concluded that there are no real solutions. Thus, we should expect that our solutions will be complex numbers.]

**SOLUTION:** First, we need to find the square of half the coefficient of the linear term (in this case, the coefficient of  $x$ ):

$$\begin{aligned}\left(\frac{1}{2} \cdot (3)\right)^2 &= \left(\frac{3}{2}\right)^2 \\ &= \frac{9}{4}\end{aligned}$$

Now add and subtract  $\frac{9}{4}$  on the left-hand side of the equation, and solve for  $x$ .

$$\begin{aligned}x^2 + 3x + 4 &= 0 \\ \Rightarrow x^2 + 3x + \frac{9}{4} - \frac{9}{4} + 4 &= 0 \\ \Rightarrow \left(x^2 + 3x + \frac{9}{4}\right) - \frac{9}{4} + 4 \cdot \frac{4}{4} &= 0 \\ \Rightarrow \left(x + \frac{3}{2}\right)\left(x + \frac{3}{2}\right) - \frac{9}{4} + \frac{16}{4} &= 0 \\ \Rightarrow \left(x + \frac{3}{2}\right)^2 + \frac{7}{4} &= 0 \\ \Rightarrow \left(x + \frac{3}{2}\right)^2 &= -\frac{7}{4} \\ \Rightarrow \sqrt{\left(x + \frac{3}{2}\right)^2} &= \pm \sqrt{-\frac{7}{4}} \\ \Rightarrow x + \frac{3}{2} &= \pm \sqrt{-1} \cdot \frac{\sqrt{7}}{2} \\ \Rightarrow x + \frac{3}{2} &= \pm i \cdot \frac{\sqrt{7}}{2} \quad (\text{since } \sqrt{-1} = i) \\ \Rightarrow x &= -\frac{3}{2} \pm \frac{\sqrt{7}}{2}i\end{aligned}$$

Thus, the solution set is  $\left\{-\frac{3}{2} + \frac{\sqrt{7}}{2}i, -\frac{3}{2} - \frac{\sqrt{7}}{2}i\right\}$ .



**Try these yourself and check your answers.**

Solve by completing the square.

**a.**  $t^2 + 12t + 25 = 0$

**b.**  $x^2 - 4x = -9$

## SOLUTIONS:

**a.**  $t^2 + 12t + 25 = 0$

$$\Rightarrow t^2 + 12t + 36 - 36 + 25 = 0 \quad (\text{add and subtract } \frac{1}{2} \cdot (12)^2 = 6^2 = 36 \text{ on the left side})$$

$$\Rightarrow (t^2 + 12t + 36) - 36 + 25 = 0$$

$$\Rightarrow (t^2 + 12t + 36) - 11 = 0$$

$$\Rightarrow (t + 6)(t + 6) - 11 = 0$$

$$\Rightarrow (t + 6)^2 - 11 + 11 = 0 + 11 \quad (\text{write as a perfect square and add the constant to both sides of the equation})$$

$$\Rightarrow (t + 6)^2 = 11$$

$$\Rightarrow \sqrt{(t + 6)^2} = \pm \sqrt{11} \quad (\text{extract the square root of both sides})$$

$$\Rightarrow t + 6 - 6 = -6 \pm \sqrt{11}$$

$$\Rightarrow t = -6 \pm \sqrt{11}$$

Thus, the solution set is  $\{-6 + \sqrt{11}, -6 - \sqrt{11}\}$ .

**b.**  $x^2 - 4x = -9$

$$\Rightarrow x^2 - 4x + 9 = -9 + 9$$

$$\Rightarrow x^2 - 4x + 9 = 0$$

$$\Rightarrow x^2 - 4x + 4 - 4 + 9 = 0 \quad (\text{add and subtract } (\frac{1}{2} \cdot (-4))^2 = (-2)^2 = 4)$$

$$\Rightarrow (x^2 - 4x + 4) - 4 + 9 = 0$$

$$\Rightarrow (x^2 - 4x + 4) + 5 = 0$$

$$\Rightarrow (x - 2)(x - 2) + 5 = 0$$

$$\Rightarrow (x - 2)^2 + 5 - 5 = 0 - 5$$

$$\Rightarrow (x - 2)^2 = -5$$

$$\Rightarrow \sqrt{(x - 2)^2} = \pm \sqrt{-5}$$

$$\Rightarrow x - 2 = \pm i\sqrt{5}$$

$$\Rightarrow x = 2 \pm i\sqrt{5}$$

Thus, the solution set is  $\{2 + i\sqrt{5}, 2 - i\sqrt{5}\}$

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