Section IV: Radical Expressions, Equations, and Functions

Module 7: The Complex Numbers

So far in your mathematics careers you have (probably) only used real numbers (denoted by $\mathbb{R}$). This set has worked pretty well for us. The only time when the real numbers have failed us is when we ran into even roots of negative numbers. In such cases we said something like, "this is not a real number."

EXAMPLE: $\sqrt{-7}$ is not a real number.

In this module we will expand our number set so that expressions like $\sqrt{-7}$ are defined. First, we need the following definition.

DEFINITION: The number $\boldsymbol{i}$ satisfies the equation $i^2 = -1$. Thus, $i = \sqrt{-1}$.

The number i enables us to simplify expressions like $\sqrt{-7}$.

EXAMPLE: Simplify the following expressions using i where appropriate.

a. $\sqrt{-7}$ **b.** $-\sqrt{-75}$

SOLUTIONS:

a.
$$\begin{aligned}\sqrt{-7} &= \sqrt{-1\cdot 7}\\ &= \sqrt{-1}\cdot\sqrt{7}\\ &= i\sqrt{7}\end{aligned}$$

b.
$$\begin{aligned}-\sqrt{-75} &= -\sqrt{25\cdot -3}\\ &= -\sqrt{25}\cdot\sqrt{-1\cdot 3}\\ &= -5i\sqrt{3}\end{aligned}$$

When we use i, we are no longer using real numbers. The number set that includes i is called the *complex numbers*.

DEFINITION: A **complex number** is a number that can be written in the form $a + bi$ where a and b are real numbers. The set of complex numbers is denoted by $\mathbb{C}$.

EXAMPLE: Which of the following are complex numbers?

a. $3 - 6i$ **b.** $4i$ **c.** -73

SOLUTIONS:

a. $3 - 6i$ is a complex number with $a = 3$ and $b = -6$.

b. $4i$ is a complex number with $a = 0$ and $b = 4$

c. -73 is a complex number with $a = -73$ and $b = 0$. Notice that this implies that **every real number is a complex numbers**!

ARITHMETIC WITH COMPLEX NUMBERS

Below, we will discuss what happens when we take powers of i as well as how to add, subtract, multiply, and divide complex numbers.

POWERS OF *i*

Using the fact that $i^2 = -1$ we can simplify powers of i.

EXAMPLE: Simplify the following.

a. i^3 **c.** i^5

b. i^4 **d.** i^6

SOLUTIONS:

a. $i^3 = i^2 \cdot i$
$= -1 \cdot i$
$= -i$

b. $i^4 = \left(i^2\right)^2$
$= (-1)^2$
$= 1$

c. $i^5 = i^4 \cdot i$
$= 1 \cdot i$
$= i$

d. $i^6 = \left(i^2\right)^3$
$= (-1)^3$
$= -1$

The example above shows us that powers of i cycle through the values i, -1, $-i$, and 1. These are the *only* values that powers of i take.

EXAMPLE: Simplify the following.

a. i^{13} **b.** i^{44}

SOLUTIONS:

a. $i^{13} = i^{12} \cdot i$
$= \left(i^2\right)^6 \cdot i$
$= (-1)^6 \cdot i$
$= 1 \cdot i$
$= i$

b. $i^{44} = \left(i^2\right)^{22}$
$= (-1)^{22}$
$= 1$

Try these yourself and check your answers.
Simplify the following.

a. i^{15} **b.** i^{100}

SOLUTIONS:

a. $$\begin{aligned} i^{15} &= i^{14} \cdot i \\ &= \left(i^2\right)^7 \cdot i \\ &= (-1)^7 \cdot i \\ &= -1 \cdot i \\ &= -i \end{aligned}$$

b. $$\begin{aligned} i^{100} &= \left(i^2\right)^{50} \\ &= (-1)^{50} \\ &= 1 \end{aligned}$$

ADDING AND SUBTRACTING COMPLEX NUMBERS

EXAMPLE: Perform the indicated operations and simplify completely.

a. $(3-8i)+(6+i)$ **b.** $(2+10i)-(16+3i)$

SOLUTIONS:

a. $(3-8i)+(6+i) = (3+6)+(-8i+i)$ (use the commutative and associative properties to combine like terms)

$= 9 + (-8+1)i$ (factor out i)

$= 9 - 7i$

b. $(2-3i)-(16-10i) = 2 - 3i - 16 + 10i$ (distribute the negative)

$= (2-16)+(-3+10)i$ (combine like terms and factor out i)

$= -14 + 7i$

Try these yourself and check your answers.
Perform the indicated operations and simplify completely.

a. $(13+8i)+(6i)$ **b.** $(7-i)-(6+12i)$

SOLUTIONS:

a. $(13+8i)+(6i) = 13+8i+6i$
$= 13+(8+6)i$
$= 13+14i$

b. $(7-i)-(6+12i) = 7-i-6-12i$
$= (7-6)+(-i-12i)$
$= 1+(-1-12)i$ (the coefficient of the term "$-i$" is -1)
$= 1-13i$

MULTIPLYING COMPLEX NUMBERS

EXAMPLE: Perform the indicated operations and simplify completely.

a. $8i \cdot 6i$ **b.** $10i \cdot (3-5i)$

SOLUTIONS:

a. $8i \cdot 6i = 8 \cdot 6 \cdot i \cdot i$
$= 48i^2$
$= -48$

b. $10i \cdot (3-5i) = 10i \cdot 3 + 10i \cdot (-5i)$ (use the distributive property)
$= 30i - 50i^2$
$= 30i + 50$
$= 50 + 30i$ (write in the form $a+bi$)

Try these yourself and check your answers.
Perform the indicated operations and simplify completely.

a. $-2i\cdot(6-i)$ **b.** $(8-3i)\cdot(10+i)$

SOLUTIONS:

a. $-2i\cdot(6-i) = -2i\cdot 6-(-2i)\cdot i$ (use the distributive property)

$= -12i + 2i^2$

$= -12i - 2$

$= -2 - 12i$ (write in the form $a + bi$)

b. $(8-3i)\cdot(10+i) = 80 - 3i\cdot 10 + 8\cdot i - 3i\cdot i$

$= 80 - 30i + 8i - 3i^2$

$= 80 - 22i + 3$

$= 83 - 22i$

DIVIDING COMPLEX NUMBERS

In order to divide complex numbers, we need to utilize *conjugates*. Recall the following definition from Section VI: Module 5:

DEFINITION: The **conjugate** of the expression $a + b$ is the expression $a - b$.

EXAMPLE: Perform the indicated operations and simplify completely.

a. $6i \div (2-3i)$ **b.** $(2-i)\div(4+5i)$

SOLUTIONS:

a. $6i \div (2-3i) = \dfrac{6i}{2-3i}$

$$= \frac{6i}{2-3i}\cdot\frac{2+3i}{2+3i}$$ (use the denominator's conjugate to rationalize the denominator)

$$= \frac{6i(2+3i)}{(2-3i)(2+3i)}$$

$$= \frac{6i\cdot 2 + 6i\cdot 3i}{2^2 + 2\cdot 3i - 2\cdot 3i - (3i)^2}$$

$$= \frac{12i + 18i^2}{4 + 6i - 6i - 9i^2}$$

$$= \frac{12i + 18\cdot -1}{4 + 0 - 9\cdot -1}$$

$$= \frac{12i - 18}{4 + 9}$$

$$= \frac{12i - 18}{13}$$

$$= -\frac{18}{13} + \frac{12}{13}i$$ (write in the form $a + bi$)

b. $(2-i) \div (4+5i) = \dfrac{2-i}{4+5i}$

$$= \frac{2-i}{4+5i}\cdot\frac{4-5i}{4-5i}$$ (use the denominator's conjugate to rationalize the denominator)

$$= \frac{(2-i)(4-5i)}{(4+5i)(4-5i)}$$

$$= \frac{8 - 10i - 4i + 5i^2}{16 - 20i + 20i - 25i^2}$$

$$= \frac{8 - 14i - 5}{16 + 25}$$

$$= \frac{3 - 14i}{41}$$

$$= \frac{3}{41} - \frac{14}{41}i$$ (write in the form $a + bi$)

Try these yourself and check your answers.
Simplify the following expressions completely.

a. $(1+3i) \div 2i$ **b.** $\dfrac{9+i}{5-4i}$

SOLUTIONS:

a.
$$(1+3i) \div 2i = \frac{1+3i}{2i}$$
$$= \frac{1+3i}{2i}\cdot\frac{i}{i}$$
(multiplying by $\frac{i}{i}$ will rationalize the denominator)
$$= \frac{(1+3i)i}{2i\cdot i}$$
$$= \frac{i+3i^2}{2i^2}$$
$$= \frac{i-3}{-2}$$
$$= \frac{3}{2} - \frac{1}{2}i$$
(write in the form $a + bi$)

b.
$$\frac{9+i}{5-4i} = \frac{9+i}{5-4i}\cdot\frac{5+4i}{5+4i}$$
(use the denominator's conjugate to rationalize the denominator)
$$= \frac{(9+i)(5+4i)}{(5-4i)(5+4i)}$$
$$= \frac{45+36i+5i+4i^2}{25+20i-20i-16i^2}$$
$$= \frac{45+41i-4}{25+16}$$
$$= \frac{41+41i}{41}$$
$$= \frac{\cancel{41}(1+i)}{\cancel{41}}$$
$$= 1+i$$