Section IV: Radical Expressions, Equations, and Functions

Module 5: Adding and Subtracting Radical Expressions

Adding and subtracting radical expressions works like adding and subtracting expressions involving variables. Just as we need *like terms* when combining expressions involving variables we need ***like radicals*** in order to combine radical expressions.

DEFINITION: Two radicals expressions are said to be **like-radicals** if they have the same indices and the same radicands.

EXAMPLE 1: **a.** The expressions $3\sqrt{5}$ and $8\sqrt{5}$ are like-radicals.

b. The expressions $3\sqrt{5}$ and $8\sqrt[4]{5}$ are **not** like radicals since they have different indices.

c. The expressions $3\sqrt{5}$ and $8\sqrt{2}$ are **not** like radicals since they have different radicands.

Since only the radicals in **a** are like, we can only combine (add and subtract) the radicals in **a**.

EXAMPLE 2: Add and subtract the pairs of radical expressions given in **EXAMPLE 1** above.

SOLUTIONS: Since only the radicals in **a** are like, we can only combine (add or subtract) the radicals in **a**.

a. ADDITION:

$$\begin{aligned} 3\sqrt{5} + 8\sqrt{5} &= (3+8)\sqrt{5} \qquad \text{(factor out } \sqrt{5}\text{)} \\ &= 11\sqrt{5} \end{aligned}$$

SUBTRACTION:

$$\begin{aligned} 3\sqrt{5} - 8\sqrt{5} &= (3-8)\sqrt{5} \qquad \text{(factor out } \sqrt{5}\text{)} \\ &= -5\sqrt{5} \end{aligned}$$

b. Neither $3\sqrt{5} + 8\sqrt[4]{5}$ nor $3\sqrt{5} - 8\sqrt[4]{5}$ can be simplified since the radicals are not like (different indices).

c. Neither $3\sqrt{5} + 8\sqrt{2}$ nor $3\sqrt{5} - 8\sqrt{2}$ can be simplified since the radicals are not like (different radicands).

Sometimes we manipulate the involved radicals so that they are like, and then combine the expressions.

EXAMPLE: Simplify the following by first obtaining like-radicals.

a. $7\sqrt[3]{16} + 4\sqrt[3]{2}$

b. $y\sqrt{98y} - \sqrt{8y^3}$

SOLUTIONS:

a. $7\sqrt[3]{16} + 4\sqrt[3]{2} = 7\sqrt[3]{8\cdot 2} + 4\sqrt[3]{2}$ (simplify $\sqrt[3]{16}$)

$= 7\sqrt[3]{8}\cdot\sqrt[3]{2} + 4\sqrt[3]{2}$

$= 7\cdot 2\cdot\sqrt[3]{2} + 4\sqrt[3]{2}$

$= 14\sqrt[3]{2} + 4\sqrt[3]{2}$

$= (14 + 4)\sqrt[3]{2}$ (factor out $\sqrt[3]{2}$)

$= 18\sqrt[3]{2}$

b. $y\sqrt{98y} - \sqrt{8y^3} = y\sqrt{49\cdot 2y} - \sqrt{4\cdot 2\cdot y^2\cdot y}$ (simplify radicals)

$= y\sqrt{49}\cdot\sqrt{2y} - \sqrt{4}\cdot\sqrt{y^2}\cdot\sqrt{2y}$

$= 7y\sqrt{2y} - 2y\sqrt{2y}$

$= (7y - 2y)\sqrt{2y}$ (factor out $\sqrt{2y}$)

$= 5y\sqrt{2y}$

When addition or subtraction is combined with *multiplication*, the distributive property is useful.

EXAMPLE: Simplify the following.

a. $\sqrt[3]{6}\left(\sqrt[3]{4} + 4\sqrt[3]{12}\right)$ **b.** $\left(\sqrt{3} + \sqrt{m}\right)\left(\sqrt{12} + \sqrt{m}\right)$

SOLUTIONS:

a. $\sqrt[3]{6}\left(\sqrt[3]{4} + 4\sqrt[3]{12}\right) = \sqrt[3]{6}\cdot\sqrt[3]{4} + \sqrt[3]{6}\cdot 4\sqrt[3]{12}$ (use the distributive property)

$= \sqrt[3]{6\cdot 4} + 4\sqrt[3]{6\cdot 12}$

$= \sqrt[3]{24} + 4\sqrt[3]{72}$

$= \sqrt[3]{8\cdot 3} + 4\sqrt[3]{8\cdot 9}$ (simplify radicals)

$= 2\sqrt[3]{3} + 4\cdot 2\sqrt[3]{9}$

$= 2\sqrt[3]{3} + 8\sqrt[3]{9}$ (these aren't like-radicals so they can't be combined)

b. $\left(\sqrt{3} + \sqrt{m}\right)\left(\sqrt{12} + \sqrt{m}\right) = \sqrt{3}\cdot\sqrt{12} + \sqrt{3}\cdot\sqrt{m} + \sqrt{m}\cdot\sqrt{12} + \sqrt{m}\cdot\sqrt{m}$

$= \sqrt{36} + \sqrt{3m} + \sqrt{12m} + \sqrt{m^2}$

$= 6 + \sqrt{3m} + \sqrt{4\cdot 3m} + m$ (simplify radicals)

$= 6 + \sqrt{3m} + 2\sqrt{3m} + m$

$= 6 + (1 + 2)\sqrt{3m} + m$ (combine like radicals)

$= 6 + 3\sqrt{3m} + m$

When adding or subtracting is combined with *division*, we need to rationalize denominators. Often, rationalizing a denominator can be accomplished by using a cleaver trick that involves the **conjugate** of the denominator.

DEFINITION: The **conjugate** of the expression $a + b$ is the expression $a - b$.

The conjugate of an expression is a related expression involving the opposite sign $(+ \text{ or } -)$. So the conjugate of expression $\sqrt{a}+\sqrt{b}$ is the expression $\sqrt{a}-\sqrt{b}$, while the conjugate of $\sqrt{a}-\sqrt{b}$ is $\sqrt{a}+\sqrt{b}$.

Conjugates are useful when rationalizing denominators since the product of two conjugates contains no radicals:

$$\begin{aligned}\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right) &= \sqrt{a}\cdot\sqrt{a}+\sqrt{b}\cdot\sqrt{a}-\sqrt{a}\cdot\sqrt{b}-\sqrt{b}\cdot\sqrt{b}\\ &= a+\sqrt{ba}-\sqrt{ab}-b\\ &= a-b\end{aligned}$$

EXAMPLE: Simplify $\dfrac{1}{\sqrt{a}-\sqrt{b}}$.

SOLUTION: If we multiply the denominator by its conjugate, we will have rationalized the denominator since the denominator will contain no radicals. In order to avoid changing the expression, we must also multiply the numerator by the conjugate of the denominator.

$$\begin{aligned}\frac{1}{\sqrt{a}-\sqrt{b}} &= \frac{1}{\sqrt{a}-\sqrt{b}}\cdot\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} \quad \text{(use the conjugate of the denominator)}\\ &= \frac{\sqrt{a}+\sqrt{b}}{\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right)}\\ &= \frac{\sqrt{a}+\sqrt{b}}{a-b}\end{aligned}$$

EXAMPLE: Simplify the following.

a. $\dfrac{6}{\sqrt{2}-3}$

b. $\dfrac{\sqrt{2}}{\sqrt{7}+\sqrt{2}}$

SOLUTIONS:

a. $$\frac{6}{\sqrt{2}-3}\cdot\frac{\sqrt{2}+3}{\sqrt{2}+3}=\frac{6\left(\sqrt{2}+3\right)}{\left(\sqrt{2}-3\right)\left(\sqrt{2}+3\right)}$$ (use the conjugate of the denominator)

$$=\frac{6\sqrt{2}+18}{\sqrt{4}+3\sqrt{2}-3\sqrt{2}-9}$$

$$=\frac{6\sqrt{2}+18}{2-9}$$

$$=\frac{6\sqrt{2}+18}{-7}$$

$$=-\frac{6\sqrt{2}+18}{7}$$

b. $$\frac{\sqrt{2}}{\sqrt{7}+\sqrt{2}}=\frac{\sqrt{2}}{\sqrt{7}+\sqrt{2}}\cdot\frac{\sqrt{7}-\sqrt{2}}{\sqrt{7}-\sqrt{2}}$$ (use the conjugate of the denominator)

$$=\frac{\sqrt{2}\left(\sqrt{7}-\sqrt{2}\right)}{\left(\sqrt{7}+\sqrt{2}\right)\left(\sqrt{7}-\sqrt{2}\right)}$$

$$=\frac{\sqrt{2}\sqrt{7}-\sqrt{2}\cdot\sqrt{2}}{\sqrt{7}\cdot\sqrt{7}-\sqrt{7}\sqrt{2}+\sqrt{2}\sqrt{7}-\sqrt{2}\cdot\sqrt{2}}$$

$$=\frac{\sqrt{14}-2}{7-\sqrt{14}+\sqrt{14}-2}$$

$$=\frac{\sqrt{14}-2}{5}$$

Try these yourself and check your answers.
Simplify the following.

a. $8\sqrt[3]{2}-\sqrt[3]{54}$ **b.** $\sqrt{10}\left(\sqrt{2}-\sqrt{5}\right)$ **c.** $\dfrac{6}{\sqrt{5}+\sqrt{3}}$

SOLUTIONS:

a. $8\sqrt[3]{2} - \sqrt[3]{54} = 8\sqrt[3]{2} - \sqrt[3]{27\cdot 2}$ (simplify $\sqrt[3]{54}$)

$= 8\sqrt[3]{2} - \sqrt[3]{27}\cdot\sqrt[3]{2}$

$= 8\sqrt[3]{2} - 3\sqrt[3]{2}$

$= (8-3)\sqrt[3]{2}$ (factor out $\sqrt[3]{2}$)

$= 5\sqrt[3]{2}$

b. $\sqrt{10}\left(\sqrt{2} - \sqrt{5}\right) = \sqrt{10}\cdot\sqrt{2} - \sqrt{10}\cdot\sqrt{5}$ (use the distributive property)

$= \sqrt{20} - \sqrt{50}$

$= \sqrt{4\cdot 5} - \sqrt{25\cdot 2}$

$= 2\sqrt{5} - 5\sqrt{2}$ (these can't be combined since the radicals aren't like)

c. $\dfrac{6}{\sqrt{5}+\sqrt{3}} = \dfrac{6}{\sqrt{5}+\sqrt{3}}\cdot\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}$ (rationalize the denominator using the conjugate)

$$= \frac{6\left(\sqrt{5}-\sqrt{3}\right)}{\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{5}-\sqrt{3}\right)}$$

$$= \frac{6\sqrt{5}-6\sqrt{3}}{\sqrt{5}\cdot\sqrt{5} - \sqrt{5}\cdot\sqrt{3} + \sqrt{3}\cdot\sqrt{5} - \sqrt{3}\cdot\sqrt{3}}$$

$$= \frac{6\sqrt{5}-6\sqrt{3}}{5 - \sqrt{15} + \sqrt{15} - 3}$$

$$= \frac{6\sqrt{5}-6\sqrt{3}}{2}$$

$= \dfrac{6\left(\sqrt{5}-\sqrt{3}\right)}{2}$ (factor out 6 from the numerator)

$= 3\left(\sqrt{5}-\sqrt{3}\right)$ (cancel the factor of 2 in the numerator and denominator)