Section IV: Radical Expressions, Equations, and Functions

Module 4: Dividing Radical Expressions

Recall the property of exponents that states that $\dfrac{a^m}{b^m} = \left(\dfrac{a}{b}\right)^m$. We can use this property to obtain an analogous property for radicals:

$$\begin{aligned} \frac{\sqrt[n]{a}}{\sqrt[n]{b}} &= \frac{a^{1/n}}{b^{1/n}} \\ &= \left(\frac{a}{b}\right)^{1/n} && \text{(using the property of exponents given above)} \\ &= \sqrt[n]{\frac{a}{b}} \end{aligned}$$

Quotient Rule for Radicals

If a and b are positive real numbers and n s a positive integer, then $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$.

EXAMPLE: Perform the indicated division, and simplify completely.

a. $\dfrac{\sqrt[3]{5}}{\sqrt[3]{40}}$ **b.** $\dfrac{\sqrt{48x^3}}{\sqrt{3x}}$

SOLUTIONS:

a.
$$\begin{aligned} \frac{\sqrt[3]{5}}{\sqrt[3]{40}} &= \sqrt[3]{\frac{5}{40}} && \text{(quotient rule for radicals)} \\ &= \sqrt[3]{\frac{1}{8}} \\ &= \sqrt[3]{\left(\frac{1}{2}\right)^3} \\ &= \frac{1}{2} \end{aligned}$$

b. $\dfrac{\sqrt{48x^3}}{\sqrt{3x}} = \sqrt{\dfrac{48x^3}{3x}}$ (quotient rule for radicals)

$= \sqrt{16x^2}$

$= \sqrt{(4x)^2}$

$= 4x$ (From now on we will assume that variables under radical signs represent non-negative numbers and omit absolute value bars.)

Quotient Rule for Simplifying Radical Expressions:

When simplifying a radical expression it is often necessary to use the following equation which is equivalent to the quotient rule:

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

EXAMPLE: Simplify the following expressions completely.

a. $\sqrt[4]{\dfrac{81}{16}}$ **b.** $\sqrt[3]{\dfrac{125}{27t^6}}$

SOLUTIONS:

a. $\sqrt[4]{\dfrac{81}{16}} = \dfrac{\sqrt[4]{81}}{\sqrt[4]{16}}$ (quotient rule for simplifying radicals)

$= \dfrac{\sqrt[4]{3^4}}{\sqrt[4]{2^4}}$ (simplify the radials in the numerator and denominator)

$= \dfrac{3}{2}$

b. $\sqrt[3]{\dfrac{125}{27t^6}} = \dfrac{\sqrt[3]{125}}{\sqrt[3]{27t^6}}$ (quotient rule for simplifying radicals)

$= \dfrac{\sqrt[3]{5^3}}{\sqrt[3]{(3t^2)^3}}$ (simplify radicals in numerator and denominator)

$= \dfrac{5}{3t^2}$

EXAMPLE: Perform the following division: $\sqrt{x} \div \sqrt[3]{x}$.

SOLUTION:

The **key step** when the indices of the radical are different is to write the expressions with rational exponents.

$$\sqrt{x} \div \sqrt[3]{x} = \frac{\sqrt{x}}{\sqrt[3]{x}}$$

$$= \frac{x^{1/2}}{x^{1/3}} \qquad \text{(write with rational exponents)}$$

$$= x^{1/2 - 1/3} \qquad \text{(use a property of exponents)}$$

$$= x^{3/6 - 2/6} \qquad \text{(create common denominator for the exponent)}$$

$$= x^{1/6}$$

$$= \sqrt[6]{x} \qquad \text{(write final answer in radical notation to agree with original expression)}$$

EXAMPLE: Perform the indicated division, and simplify completely.

a. $\dfrac{\sqrt[5]{t^4}}{\sqrt{t}}$ **b.** $\dfrac{\sqrt[6]{a^3b^5}}{\sqrt[3]{ab^2}}$

SOLUTIONS:

a. $$\frac{\sqrt[5]{t^4}}{\sqrt{t}} = \frac{t^{4/5}}{t^{1/2}} \qquad \text{(write with rational exponents)}$$

$$= t^{4/5 - 1/2} \qquad \text{(use a property of exponents)}$$

$$= t^{8/10 - 5/10} \qquad \text{(create a common denominator for the exponent)}$$

$$= t^{3/10}$$

$$= \sqrt[10]{t^3} \qquad \text{(write final answer in radical notation to agree with original expression)}$$

b. $\dfrac{\sqrt[6]{a^3b^5}}{\sqrt[3]{ab^2}} = \dfrac{\left(a^3b^5\right)^{1/6}}{\left(ab^2\right)^{1/3}}$ (write with rational exponents)

$= \dfrac{a^{3/6}b^{5/6}}{a^{1/3}b^{2/3}}$ (use a property of exponents)

$= a^{3/6\,-\,1/3} \cdot b^{5/6\,-\,2/3}$ (use another property of exponents)

$= a^{3/6\,-\,2/6} \cdot b^{5/6\,-\,4/6}$ (create common denominator for the exponents)

$= a^{1/6}b^{1/6}$

$= (ab)^{1/6}$ (use another property of exponents)

$= \sqrt[6]{ab}$ (write final answer in radical notation to agree with the original expression)

RATIONALIZING DENOMINATORS

[Recall from Section I: Module 2 that the set of **rational numbers** consists of all numbers that can be expressed as the ratio of integers. In other words, a rational number can be expressed as a fraction where the numerator and denominator are both whole numbers. Although it's true that there are many, many different fractions ($\frac{1}{2}$, $\frac{3}{5}$, $-\frac{13}{7}$, $\frac{9}{4}$, etc.), there are many, many, many more numbers that **cannot** be expressed as fractions. Numbers that cannot be expressed as a ratio of integers are called **irrational numbers**. (You may be familiar with one of the characteristics of irrational numbers: their decimal expansions never end and never repeat.) The number π is probably the most famous irrational number, but there are lots of others – actually, there are infinitely many! Most radical expresses are irrational numbers, e.g., numbers like $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt[3]{13}$, $\sqrt[4]{21}$, and $\sqrt[7]{43}$ are irrational.]

Before the 1970’s there were no electronic handheld calculators so mathematicians, scientists, and students of mathematics needed to consult previously created tables to obtain approximations of calculations. In order to minimize the number of tables that were needed, all calculations were made by using numbers whose denominators were **rational numbers**. Thus, in the "old-days", it was important to be able to *rationalize denominators*. so you could then look at a table and get an approximation. But as the *Collector's Guide to Pocket Calculators* states, “1971 heralded the age of the low-cost consumer handheld calculator.” Nowadays, we all have calculators that can readily give us highly accurate approximations of calculations. Although we no longer need to rationalize denominators in order to obtain approximations of calculations, the skill we learn in this section is an important algebraic manipulation used in calculus and beyond.

Since radical expressions are often irrational, we study rationalizing denominators while we are focusing on radical expressions. In this context, rationalizing denominators consists of getting all of the radicals out of the denominator of the expression.

EXAMPLE: Rationalize the denominators in the following expressions:

a. $\dfrac{5}{\sqrt{3}}$ **b.** $\dfrac{8}{\sqrt[3]{2}}$ **c.** $\dfrac{11}{\sqrt[5]{x^2}}$

SOLUTIONS:

a. $$\frac{5}{\sqrt{3}} = \frac{5}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$ (obtain a perfect square under the square root in the denominator)

$$= \frac{5\sqrt{3}}{\sqrt{3^2}}$$

$$= \frac{5\sqrt{3}}{3}$$

b. $$\frac{8}{\sqrt[3]{2}} = \frac{8}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{2^2}}{\sqrt[3]{2^2}}$$ (obtain a perfect cube under the cube root in the denominator)

$$= \frac{8\sqrt[3]{4}}{\sqrt[3]{2^3}}$$

$$= \frac{8\sqrt[3]{4}}{2}$$

c. $$\frac{11}{\sqrt[5]{x^2}} = \frac{11}{\sqrt[5]{x^2}} \cdot \frac{\sqrt[5]{x^3}}{\sqrt[5]{x^3}}$$ (obtain a power-of-five under the fifth root in the denominator)

$$= \frac{11\sqrt[5]{x^3}}{\sqrt[5]{x^5}}$$

$$= \frac{11\sqrt[5]{x^3}}{x}$$