Section IV: Radical Expressions, Equations, and Functions

Module 2: Rational Numbers as Exponents

In Introductory Algebra (MTH 60/65 at PCC) and in Section I, Module 1: Review you should have studied how to work with exponents that are integers. In this module we will study how to work with exponents that are rational numbers (i.e., fractions).

EXAMPLE: Simplify the expression $9^{1/2}$.

SOLUTION: Since we want the properties of exponents to apply to rational exponents, we see that the square of $9^{1/2}$ must be 9:

$$\begin{aligned}\left(9^{1/2}\right)^2 &= 9^{(1/2)\cdot 2}\\ &= 9^1\\ &= 9\end{aligned}$$

Since the square of $9^{1/2}$ is 9, we see that $9^{1/2}$ is the square root of 9:

$$\begin{aligned}9^{1/2} &= \sqrt{9}\\ &= 3\end{aligned}$$

Rational Exponents and Radical Notation:

For all positive integers n, $a^{1/n} = \sqrt[n]{a}$.

EXAMPLE: Write the following expressions in radical notation.

a. $t^{1/7}$ **b.** $(2xy)^{1/3}$

SOLUTIONS:

a. $t^{1/7} = \sqrt[7]{t}$

b. $(2xy)^{1/3} = \sqrt[3]{2xy}$

EXAMPLE: Simplify the expression $8^{2/3}$.

SOLUTION: We can use the properties of exponents (see Section I, Module 1: Review) to write $8^{2/3}$ as $\left(8^{1/3}\right)^2$, which we know how to simplify:

$$\begin{aligned} 8^{2/3} &= \left(8^{1/3}\right)^2 \\ &= \left(\sqrt[3]{8}\right)^2 \\ &= 2^2 \\ &= 4 \end{aligned}$$

Of course we can also write $8^{2/3}$ as $\left(8^2\right)^{1/3}$, which we can simplify:

$$\begin{aligned} 8^{2/3} &= \left(8^2\right)^{1/3} \\ &= (64)^{1/3} \\ &= \sqrt[3]{64} \\ &= 4 \end{aligned}$$

This example suggests the following important fact:

Rational Exponents and Radical Notation:

For all positive integers m and n, $a^{m/n} = \sqrt[n]{a^m}$ and $a^{m/n} = \left(\sqrt[n]{a}\right)^m$.

EXAMPLE: Simplify the following expressions.

a. $27^{4/3}$ **b.** $32^{2/5}$

SOLUTIONS:

a.
$$\begin{aligned} 27^{4/3} &= \left(27^{1/3}\right)^4 \\ &= \left(\sqrt[3]{27}\right)^4 \\ &= 3^4 \\ &= 81 \end{aligned}$$

b.
$$\begin{aligned} 32^{2/5} &= \left(32^{1/5}\right)^2 \\ &= \left(\sqrt[5]{32}\right)^2 \\ &= 2^2 \\ &= 4 \end{aligned}$$

EXAMPLE: Simplify the expression $\left(4x^{4/3}y^{1/6}\right)^{3/2}$.

SOLUTION: Since rational exponents are exponents, we can use the properties of exponents to simplify the expression. (see Section I, Module 1: Review)

$$\begin{aligned}\left(4x^{4/3}y^{1/6}\right)^{3/2} &= 4^{3/2}\left(x^{4/3}\right)^{3/2}\left(y^{1/6}\right)^{3/2}\\ &= \left(4^{1/2}\right)^{3} x^{(4/3)\cdot(3/2)}y^{(1/6)\cdot(3/2)} \qquad \text{(multiply exponents)}\\ &= \left(\sqrt{4}\right)^{3} x^{12/6}y^{3/12}\\ &= 2^{3}x^{2}y^{1/4}\\ &= 8x^{2}y^{1/4}\end{aligned}$$

EXAMPLE: Simplify the following expressions.

a. $\left(a^2b^3\right)^{1/6}$ **b.** $\left(\dfrac{p^{1/4}}{p^{3/4}}\right)^{8}$

SOLUTIONS:

a. $$\begin{aligned}\left(a^2b^3\right)^{1/6} &= \left(a^2\right)^{1/6}\left(b^3\right)^{1/6}\\ &= a^{2/6}b^{3/6}\\ &= a^{1/3}b^{1/2}\end{aligned}$$

b. $$\begin{aligned}\left(\frac{p^{1/4}}{p^{3/4}}\right)^{8} &= \left(p^{1/4\,-\,3/4}\right)^{8}\\ &= \left(p^{-2/4}\right)^{8}\\ &= \left(p^{-1/2}\right)^{8}\\ &= p^{-4}\\ &= \frac{1}{p^4}\end{aligned}$$

EXAMPLE: Simplify the expression $\sqrt[4]{x^3\sqrt{x^3}}$ completely.

SOLUTION: When an expression contains more than one radical, it's best to first convert to rational exponents:

$$\begin{aligned}
\sqrt[4]{x^3\sqrt{x^3}} &= \left(x^3 \cdot x^{3/2}\right)^{1/4} \\
&= \left(x^3\right)^{1/4} . \left(x^{3/2}\right)^{1/4} \\
&= x^{3/4} \cdot x^{3/8} \\
&= x^{6/8} \cdot x^{3/8} \\
&= x^{9/8} \\
&= x^{8/8} \cdot x^{1/8} \\
&= x\sqrt[8]{x}
\end{aligned}$$

Note that the last two lines aren't necessary unless the directions specify that answers must be left in "radical form." But if there aren't specific directions, it's usually best to leave the answer in the same form that the original expression has. Here, since the original is in radical from, I've left the answer in radical form, but if the original had exponents instead, my answer should have involved exponents. Also note that this isn't the only way to do this problem. Instead, you could combine the exponents "3" and "3/2" in the third line before distributing the exponent "1/4".

Try this yourself and check your answer.

Simplify the expression $\sqrt{x^2 \cdot \sqrt[5]{x^4}}$ completely.

SOLUTION:

$$\begin{aligned}
\sqrt{x^2 \cdot \sqrt[5]{x^4}} &= \left(x^2 \cdot x^{4/5}\right)^{1/2} \\
&= \left(x^{10/5} \cdot x^{4/5}\right)^{1/2} \\
&= \left(x^{14/5}\right)^{1/2} \\
&= x^{7/5} \\
&= x^{5/5 + 2/5} \\
&= x^{5/5}x^{2/5} \\
&= x \cdot \sqrt[5]{x^2}
\end{aligned}$$