Section II: Functions, Inequalities, and the Absolute Value

Module 3: The Absolute Value

The **absolute value function** is defined as follows:

$$f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

The graph of $f(x) = |x|$ is shown in Figure 1 below.

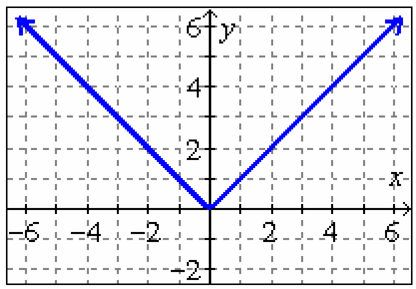


Figure 1: $f(x) = |x|$

EXAMPLE: Solve $|x| = 7$.

SOLUTION: Since both $|7| = 7$ and $|-7| = 7$, both 7 and -7 are solutions, so the solution set is $\{7, -7\}$.

THE ABSOLUTE-VALUE PRINCIPLE FOR EQUATIONS

For any positive number a and any algebraic expression X:

- if $a > 0$ then the equation $|X| = a$ has two solutions.
- if $a = 0$ then the equation $|X| = a$ has one solution.
- if $a < 0$ then the equation $|X| = a$ has no solutions.

EXAMPLE: Solve $\left|t-5\right|=3$.

SOLUTION: Removing the absolute value symbols yields

$$t-5=3 \quad \text{or} \quad t-5=-3$$

which implies that

$$t=8 \quad \text{or} \quad t=2.$$

Thus, the solution set is $\{8, 2\}$.

EXAMPLE: Solve $\left|z+10\right|=0$.

SOLUTION: In order that $\left|z+10\right|=0$ we need $z+10=0$. Thus, $z=-10$ and the solution set is $\{-10\}$.

EXAMPLE: Solve $\left|x-5\right|<3$ graphically.

SOLUTION: Below, we've graphed $y=\left|x-5\right|$ and $y=3$ (i.e., the left and right sides of the inequality). Since the absolute value function is below the horizontal line $y=3$ when $2<x<8$, the solution set for $\left|x-5\right|<3$ is $\{x \mid x\in\mathbb{R} \text{ and } 2<x<8\}$ which can be written in interval notation as $(2, 8)$. Another way to say this is that the y-values (i.e., the output values) of $y=\left|x-5\right|$ are less than 3 when $2<x<8$, so the solution set for $\left|x-5\right|<3$ is the interval $(2, 8)$.

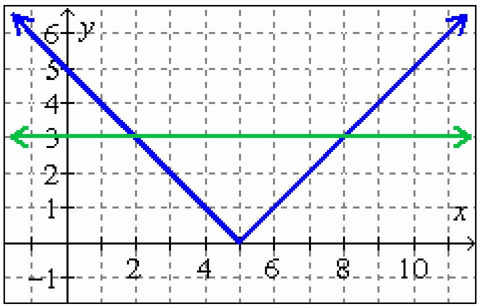


Figure 2: $y=\left|x-5\right|$ and $y=3$

PRINCIPLES FOR SOLVING ABSOLUTE VALUE PROBLEMS

For any positive number a and any algebraic expression X:

- the solutions of $|X| = a$ are those numbers that satisfy $X = -a$ or $X = a$.
- the solutions of $|X| < a$ are those numbers that satisfy $-a < X < a$.
- the solutions of $|X| > a$ are those numbers that satisfy $X < -a$ or $X > a$.

EXAMPLE: Solve $|x + 2| > 4$.

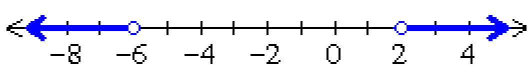
SOLUTION: Removing the absolute value symbols yields

$$x + 2 < -4 \text{ or } x + 2 > 4,$$

which can be simplified to

$$x < -6 \text{ or } x > 2.$$

Thus, the solution set for $|x + 2| > 4$ is $(-\infty, -6) \cup (2, \infty)$. We've graphed this set on the number line below.



Try these yourself and check your answers.

a. Solve $|2x - 3| = 5$.

b. Solve $|p| = -5$.

c. Solve $|3x - 2| < 7$.

SOLUTIONS:

a.

$$
\begin{aligned}
& & |2x - 3| &= 5 \\
&\Rightarrow & 2x - 3 = 5 &\text{ or } 2x - 3 = -5 \\
&\Rightarrow & 2x - 3 + 3 = 5 + 3 &\text{ or } 2x - 3 + 3 = -5 + 3 \\
&\Rightarrow & 2x = 8 &\text{ or } 2x = -2 \\
&\Rightarrow & \frac{2x}{2} = \frac{8}{2} &\text{ or } \frac{2x}{2} = \frac{-2}{2} \\
&\Rightarrow & x = 4 &\text{ or } x = -1
\end{aligned}
$$

Thus, the solution set is $\{4, -1\}$.

b. The equation $|p| = -5$ has no solutions since the absolute value of a number is never negative.

c.

$$
\begin{aligned}
& & |3x - 2| &< 7 \\
&\Rightarrow & -7 < 3x - 2 &< 7 \\
&\Rightarrow & -7 + 2 < 3x - 2 + 2 &< 7 + 2 \\
&\Rightarrow & -5 < 3x &< 9 \\
&\Rightarrow & \frac{-5}{3} < \frac{3x}{3} &< \frac{9}{3} \\
&\Rightarrow & -\frac{5}{3} < x &< 3
\end{aligned}
$$

Thus, the solution set is $\left(-\frac{5}{3}, 3\right)$.