

Section II: Functions, Inequalities, and the Absolute Value

Module 1: Introduction to Functions

A *function* is a special type of *binary relation*. So before we discuss what a function is, we need to define *binary relation*.



DEFINITION: A **binary relation** is a *rule* that establishes a relationship between *two* sets. (Note that this is an informal definition.)

The two sets involved in a binary relation play different roles; these roles are determined by the rule of the relation (see the first example). The two roles that the sets play are the “first set” or the “set of inputs” and the “second set” or the “set of outputs.” The set of inputs is called the **domain**; the set of outputs is called the **range**.



- EXAMPLE:** a. If the rule of the relation is “Associate each person with his/her social security number” then the domain is the set of people and the range is the set of social security numbers.
- b. If the rule of the relation is “Associate each social security number with the person who has been assigned to that number,” then the domain is the set of social security numbers and the range is the set of people.

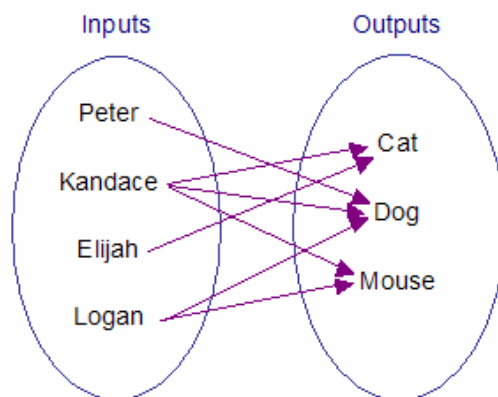


EXAMPLE: Consider a *binary relation* defined on the following two sets:

Set of inputs (domain): {Peter, Kandace, Elijah, Logan}

Set of outputs (range): {Cat, Dog, Mouse}

The *rule* that relates these sets is that “each person (in the set of inputs) is related to an animal (from the set of outputs) that he or she has as a pet.” The **arrow diagram** below defines this relation:



The arrow diagram tells us, for example that “Logan is related to Dog” (since there is an arrow from Logan to Dog), and this means that Logan has a dog. We also see that “Logan is related to Mouse,” so Logan also has a mouse.

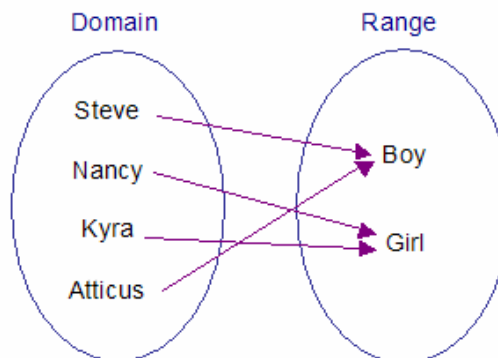
Another way to convey the information contained in a relation is to use **ordered pairs**. To represent the fact that “Logan is related to Dog” we use the ordered pair (Logan, Dog). The first term in each ordered pair is an element of the set of inputs and the second term in each ordered pair is an element of the set of outputs. The entire relation represented by the arrow diagram above can be represented by the set of ordered pairs below:

$$\{(Peter, Dog), (Kandace, Cat), (Kandace, Dog), (Kandace, Mouse), (Elijah, Cat), (Logan, Dog), (Logan, Mouse)\}$$


DEFINITION: A **function** is a binary relation in which each element in the domain corresponds to *exactly one* element in the range.



EXAMPLE: The arrow diagram below represents a binary relation on the sets {Steve, Nancy, Kyra, Atticus} and {Boy, Girl}. The “rule” that defines this relation is “Associate each person with his/her gender.” This relation is a *function* since each person has exactly one gender. On the arrow diagram we can tell that this is a *function* since there is exactly one arrow coming out of each element in the domain. (Notice that in the example above, more than one arrow comes from some elements in the domain, so the diagram does not represent a function.)



We can represent this function by the following set of ordered pairs:

$$\{(Steve, Boy), (Nancy, Girl), (Kyra, Girl), (Atticus, Boy)\}$$



EXAMPLE: Determine whether each of the following relations describes a function with domain $\{1, 2, 3, 4\}$ and range $\{5, 6, 7, 8\}$.

a. $\{(1, 5), (2, 6), (3, 7), (4, 8)\}$

b. $\{(1, 7), (1, 8), (2, 5), (3, 6)\}$

c. $\{(1, 5), (2, 5), (3, 5), (4, 5)\}$

SOLUTIONS:

a. $\{(1, 5), (2, 6), (3, 7), (4, 8)\}$ represents a function because each element in the domain corresponds to *exactly one* element in the range.

b. $\{(1, 7), (1, 8), (2, 5), (3, 6)\}$ does not represent a function because the number 1 corresponds to *two different* elements of the range.

c. $\{(1, 5), (2, 5), (3, 5), (4, 5)\}$ is a function because each element in the domain corresponds to *exactly one* element in the range. (Note that it is okay to repeat elements in the range.)



EXAMPLE: The table below defines the function f , where the inputs come from $A = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$ and the outputs come from $B = \{-5, -3, 0, 1, 2, 3, 4, 9, 13\}$. Thus, the set A is the **domain** of f while the set B is the **range** of f , which we can represent with the notation $f : A \rightarrow B$.

DOMAIN OF f	-5	-4	-3	-2	-1	0	1	2	3	4	5
RANGE OF f	4	-3	-5	4	1	3	0	9	-3	2	13

If someone were to ask you, “Where does f send -2 ?” you would (hopefully) say, “ f sends -2 to 4” (since there is a column in the table showing that the element -2 in the domain is related to 4 in the range). **Function notation** offers another way to communicate this information:

$$f(-2) = 4 \quad \text{means} \quad “f \text{ sends } -2 \text{ to } 4.”$$

Use the table above to evaluate the following:

a. Evaluate $f(-4)$.



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FOR ANSWER](#)

b. Evaluate $f(1)$.



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c. Evaluate $f(9)$.



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Sometimes we don't know what the input is. In such a case, we can call the input x (or any other variable), where x represents a generic element of the domain; then $f(x)$ represents the element in the range that x is associated with under the function f . In this example, the domain of f is the set A and the range is the set B . Thus, $x \in A$ and $f(x) \in B$. With this new notation, we can rename the rows in our table:

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$f(x)$	4	-3	-5	4	1	3	0	9	-3	2	13

Solve the following for x .

d. Solve $f(x) = 1$.



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e. Solve $f(x) = -3$.



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f. Solve $f(x) = 5$.



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EQUATIONS AS FUNCTIONS

Functions are often defined by equations. For example, the equation $y = x + 3$ defines a function, where the inputs are represented by x and the outputs are represented by y . It is often helpful to write equations like $y = x + 3$ in function notation:

$$y = x + 3 \Rightarrow f(x) = x + 3$$

which is read " f of x equals x plus 3".

Similarly, the equation $u = 3t - 5$ can be written in function notation as $g(t) = 3t - 5$. This is read: " g of t equals $3t$ minus 5."

Caution: The notation $f(x)$ does not mean multiplication of f by x .

Function notation involving $f(x)$ is more compact than notation involving " $y =$ ". For example:

With function notation: If $f(x) = x + 3$, then $f(5) = 8$.

Without function notation: If $y = x + 3$, then the value of y is 8 when x is 5.

The **independent variable** refers to the variable representing possible values in the domain, and the **dependent variable** refers to the variable representing possible values in the range. Thus in our usual ordered pair notation (x, y) , x is the independent variable and y is the dependent variable. It is customary to plot the independent variable on the horizontal axis and the dependent variable on the vertical axis.



EXAMPLE: Find the indicated function value.

a. $f(7)$ if $f(x) = 3x - 5$

b. $g(-3)$ if $g(x) = x^2 - x + 4$

c. $h(-1)$ if $h(t) = -t + 5$

d. $k(0)$ if $k(m) = m^{50} + 13m^{36} - 62m^4 + 6$

SOLUTIONS:

a. $f(7) = 3(7) - 5$
 $= 21 - 5$
 $= 16$

b. $g(-3) = (-3)^2 - (-3) + 4$
 $= 9 + 3 + 4$
 $= 16$

c. $h(-1) = -(-1) + 5$
 $= 1 + 5$
 $= 6$

d. $k(0) = (0)^{50} + 13(0)^{36} - 62(0)^4 + 6$
 $= 6$



EXAMPLE: Suppose $f(x) = x^2 - x$. Find

a. $f(2a)$.

b. $f(a + b)$.

c. $f(a) + b$.

SOLUTIONS:

$$\begin{aligned}\text{a. } f(2a) &= (2a)^2 - 2a \\ &= 4a^2 - 2a\end{aligned}$$

$$\begin{aligned}\text{b. } f(a + b) &= (a + b)^2 - (a + b) \\ &= (a + b)(a + b) - a - b \\ &= a^2 + 2ab + b^2 - a - b\end{aligned}$$

$$\begin{aligned}\text{c. } f(a) + b &= (a)^2 - a + b \\ &= a^2 - a + b\end{aligned}$$



EXAMPLE: Suppose $g(x) = 3x$. Find

$$\text{a. } g(x + 2).$$

$$\text{b. } g(x) + 2.$$

$$\text{c. } g(2x).$$

$$\text{d. } g(2x - 2).$$

SOLUTIONS:

$$\begin{aligned}\text{a. } g(x + 2) &= 3(x + 2) \\ &= 3x + 6\end{aligned}$$

$$\text{b. } g(x) + 2 = 3x + 2$$

$$\begin{aligned}\text{c. } g(2x) &= 3(2x) \\ &= 6x\end{aligned}$$

$$\begin{aligned}\text{d. } g(2x - 2) &= 3(2x - 2) \\ &= 6x - 6\end{aligned}$$



EXAMPLE: The function f is graphed in Figure 1 below. (Note that there is an arrow on the left-hand side of the graph since the function doesn't end when the graph ends. If the graph ends, it will be denoted with a filled-in circle, as there is on the right-hand side.) Determine the following:

- a. $f(1)$
- b. the domain of f
- c. any x -value(s) for which $f(x) = 2$
- d. any x -value(s) for which $f(x) = -1$
- e. the range of f

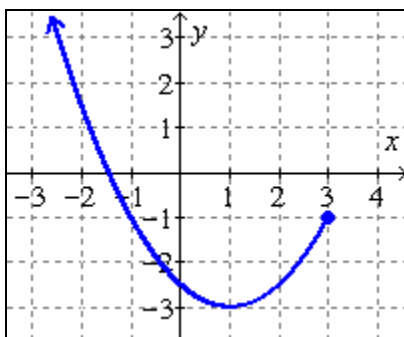


Figure 1: Graph of $y = f(x)$.

SOLUTIONS:

- a. $f(1) = -3$ which can be determined because the point $(1, -3)$ is on the graph.
- b. The domain of f is $(-\infty, 3]$ since the graph suggests that there is a point above or below every x -value less than or equal to 3; every x -value less than or equal to 3 is "used" by the function so it's in the domain.
- c. $f(x) = 2$ when $x \approx -2.2$ because $f(-2.2) \approx 2$. (Notice that we use " \approx " to denote the fact that we are *approximating* the answer.)
- d. $f(x) = -1$ when $x = -1$ or $x = 3$.
- e. The range of f is $[-3, \infty)$ since the graph suggests that there is a point to the left or right of every y -value greater than or equal to -3 ; every y -value greater than or equal to -3 is "used" by the function so it's in the range.



EXAMPLE: The function p is graphed in Figure 2 below. Find the following:

- $p(1)$.
- the domain of p .
- all x -values for which $p(x) = 2$.
- the range of p .

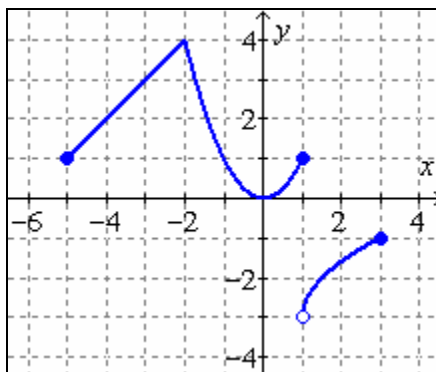


Figure 2: Graph of $y = p(x)$.

SOLUTIONS:

- $p(1) = 1$ because the point $(1, 1)$ is on the graph. Note that there is a hole at the point $(1, -3)$, so the point $(1, -3)$ is NOT on the graph.
- The domain of p is $[-5, 3]$ (in interval notation) or $\{x \mid x \in \mathbb{R} \text{ and } -5 \leq x \leq 3\}$ (in set-builder notation).
- The x -values for which $p(x) = 2$ are $x = -4$ and $x \approx -1.5$.
- The range of p is $(-3, -1] \cup [0, 4]$ (in interval notation).

Recall that the definition of a function requires that each input has *only one* corresponding output. Since the inputs are x -values and the outputs are y -values, a function can have only one y -value for each x -value. If we translate this information to the context of the *graph* of a function, we quickly realize that (since each x -value is represented by a vertical line) the graph of a function can have at most one point on each vertical line. (If a vertical line $x = a$ passes through a graph more than one time, then the x -value a has more than one corresponding output, so the graph cannot represent a function.) This leads us to the *Vertical Line Test*.

The Vertical-Line Test

If you graph a relation on the xy -plane and every vertical line intersects the graph at most once, then y is a function of x . (If it is possible to draw a vertical line that intersects the graph more than once, then y is not a function of x .)



EXAMPLE: In which of the following graphs is y a function of x ?

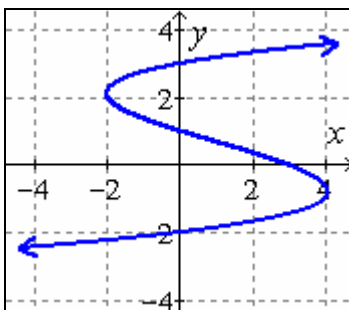


Figure 3

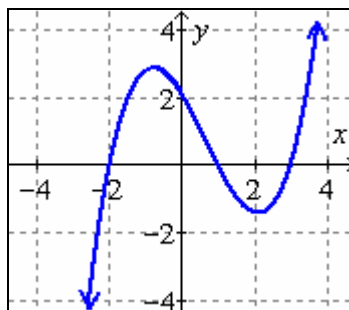


Figure 4

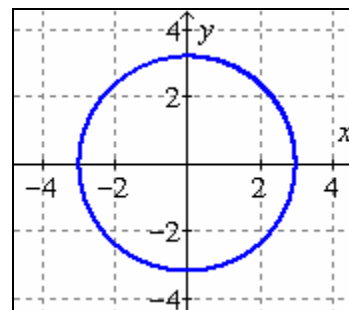


Figure 5

SOLUTION: In Figures 3 and 5, it is possible to draw vertical lines that intersect the graphs more than once (e.g., the y -axis is a vertical line that intersects these graphs more than once), so y is **not** a function of x in these graphs. In Figure 4, the graph passes the vertical line test since no vertical line intersects the graph more than once, so y a function of x in this graph.



EXAMPLE: The function $y = g(x)$ is defined in Figure 6, the function $y = f(x)$ is defined in Table 1, and $h(x) = x^2 + 1$.

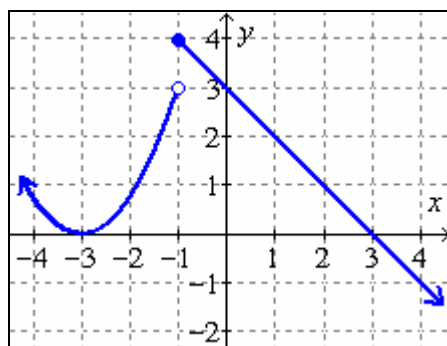


Figure 6: Graph of $y = g(x)$

Table 1

x	$f(x)$
-2	3
-1	5
0	-2
1	-1
2	7

Find each of the following:

- | | | | |
|------------|--------------|----------------------|----------------------|
| a. $f(1)$ | c. $h(-3)$ | e. $h(2x + 3)$ | g. x if $g(x) = 0$ |
| b. $g(-1)$ | d. $f(g(4))$ | f. x if $f(x) = 7$ | h. x if $h(x) = 5$ |

SOLUTIONS:

a. $f(1) = -1$ (using the table)

b. $g(-1) = 4$ (using the graph)

Note that $g(-1) \neq 3$ since $(-1, 3)$ is not a point on the graph. The solid circles represent points on the graph, whereas the open circle at $(-1, 3)$ represents a hole in the graph.

c. $h(-3) = (-3)^2 + 1$
 $= 9 + 1$
 $= 10$

d. $f(g(4)) = f(-1)$ (first calculate $g(4)$ using the graph)
 $= 5$ (then calculate $f(-1)$ from the table)

e. $h(2x + 3) = (2x + 3)^2 + 1$ (the input is the entire expression $2x + 3$)
 $= (2x + 3)(2x + 3) + 1$
 $= 4x^2 + 6x + 6x + 9 + 1$
 $= 4x^2 + 12x + 10$

f. Since $f(2) = 7$, if $f(x) = 7$, then $x = 2$. Look for an output value of 7 in the table.

g. To solve this equation we need to find an output value of 0 on the graph of $y = g(x)$. (So we are looking for the x -intercepts on the graph of $y = g(x)$). Since $g(-3) = 0$ and $g(3) = 0$, if $g(x) = 0$, then $x = -3$ or $x = 3$. Another way of writing this using set notation is $x \in \{-3, 3\}$, or we could say, "The solution set for the equation $g(x) = 0$ is $\{-3, 3\}$."

h. $h(x) = 5$
 $\Rightarrow x^2 + 1 = 5$
 $\Rightarrow x^2 = 4$
 $\Rightarrow x = 2$ or $x = -2$
