Section III: Rational Expressions, Equations, and Functions

Module 6: Applications Involving Rational Equations

When solving real-world problems we occasionally come across rational equations. In this module we will study two applications that require us to solve rational equations: *applications involving work* and *applications involving motion*.

APPLICATIONS INVOLVING WORK

EXAMPLE: Steve and Janet are going to paint the picket fence that surrounds their house today. Steve can paint the fence alone in 12 hours. Janet can paint the fence alone in 9 hours. How long will it take them to paint the fence together?

SOLUTION:

Be careful! We cannot assume that Steve and Janet each paint half of the fence. Since it takes Janet less time than Steve to paint the fence alone, she paints at a faster rate. So when she paints with Steve, she will paint more of the fence than does Steve. Thus, *we must not assume that each person paints half of the fence*.

To solve this problem, we should first **define a variable** that represents the quantity that we are looking for:

Let t represent the time, in hours, that it takes Steve and Janet to paint the fence together.

To find an equation that represents the given scenario, let's determine how much of the fence Steve and Janet paint, respectively.

Since it takes Steve 12 hours to paint the fence, each hour he paints $\frac{1}{12}$ of the fence.

Since it takes them t hours to paint the fence together, Steve paints $\frac{1}{12}t$ of the fence.

Since it takes Janet 9 hours to paint the fence, each hour she paints $\frac{1}{9}$ of the fence.

Since it takes them t hours to paint the fence together, Janet paints $\frac{1}{9}t$ of the fence.

Since the $\frac{1}{12}t$ of the fence that Steve paints and the $\frac{1}{9}t$ of the fence that Janet paints together constitute one complete fence, we obtain the equation

$$\frac{1}{12}t + \frac{1}{9}t = 1 \quad \text{or} \quad \frac{t}{12} + \frac{t}{9} = 1.$$

Let's solve this equation for t:

$$
\begin{aligned}
& \frac{t}{12} + \frac{t}{9} = 1 \\
\Rightarrow\quad & 36 \cdot \left(\frac{t}{12} + \frac{t}{9} \right) = 1 \cdot 36 \qquad \text{(multiply both sides by the LCD)} \\
\Rightarrow\quad & 3t + 4t = 36 \\
\Rightarrow\quad & 7t = 36 \\
\Rightarrow\quad & t = \tfrac{36}{7} \approx 5.15
\end{aligned}
$$

(Be sure to check the solution!)

Thus, (since the solutions checks and since 0.15 hours equals 9 minutes) it will take Steve and Janet about 5 hours and 9 minutes to paint the fence together.

EXAMPLE: After painting the fence Steve and Janet realize that their house also needs painting. But Steve will have to do the painting himself since Janet has injured her shoulder. Before getting started, Steve wants to find out how long the project will take. He determines that it would take them 9 days to paint the house together (if they were both healthy) and that it would take Janet (when healthy) 16 days to paint the house alone. How long will it take Steve to paint the house alone?

SOLUTION:

To solve this problem, we should first **define a variable** that represents the quantity that we are looking for:

Let d represent the number of days that it will take Steve to paint the house alone.

Since it takes Steve d days to paint the house, each day he paints $\frac{1}{d}$ of the house. Since it takes them 9 days to paint the house together, Steve would paint $\frac{9}{d}$ of the house if he and Janet painted it together. Since it takes Janet 16 days to paint the house alone, each day she paints $\frac{1}{16}$ of the house. Since it takes them 9 days to paint the house together, Janet would paint $\frac{9}{16}$ of the house if she and Steve painted it together.

If they painted together, the $\frac{9}{d}$ of the house that Steve paints and the $\frac{9}{16}$ of the house that Janet paints would together constitute one complete house, so we obtain the equation

$$\frac{9}{d} + \frac{9}{16} = 1 .$$

Let's solve this equation for d:

$$\frac{9}{d} + \frac{9}{16} = 1$$

$$\Rightarrow 16d \cdot \left(\frac{9}{d} + \frac{9}{16}\right) = 1 \cdot 16d \qquad \text{(multiply both sides by the LCD)}$$

$$\Rightarrow \quad 16 \cdot 9 + 9d = 16d$$

$$\Rightarrow \quad 144 + 9d = 16d$$

$$\Rightarrow \quad 144 = 7d$$

$$\Rightarrow \quad d = \tfrac{144}{7} \approx 20.57$$

(Be sure to check the solution!)

So (since the solutions checks) it will take Steve a little more than 20.5 days to paint the house alone.

Below we summarize the method used in the examples above to obtain our equations.

Solving Applications Involving Work

If people M and N work together on a project, then

$$\frac{t}{m} + \frac{t}{n} = 1$$

where

m is the time needed for person M to complete the project,
n is the time need for person N to complete the project, and
t is the time needed for M and N to complete the project together.

APPLICATIONS INVOLVING MOTION

In order to solve problems that involve things that move we often need to utilize a derivation of the famous formula $d = r \cdot t$: *distance* (d) *equals the rate* (r) *times the time* (t). If we solve for time (t) we obtain the formula $t = \frac{d}{r}$ or if we solve for rate (r) we obtain the formula $r = \frac{d}{t}$.

EXAMPLE: I went canoeing on the Winding River last Saturday. I traveled 15 miles downstream and then turned around and traveled the same 15 miles upstream. The trip took me a total of 4 hours. If my canoe travels 8 mph (miles per hour) in still water, what is the speed of the current in the river?

SOLUTION:

Since the total time spent canoeing was 4 hours, we can obtain an equation by adding the time spent canoeing upstream with the time spent canoeing downstream and setting this sum equal to 4. But before we do anything, we need to define a variable. Since we are asked to find the speed of the current in the Winding River, let's **define a variable** to represent this speed:

Let s represent the speed of the river (in mph).

Since we want to add the times spent canoeing up and downstream, we will want to utilize the formula $t = \frac{d}{r}$ to obtain expressions for the upstream and downstream times.

When I canoed downstream, I traveled 15 miles (so $d = 15$) and the rate that I traveled was $r = 8 + s$ mph (the sum of my speed in still water and the rate of the current). So the time spent traveling downstream (t_1) was

$$t_1 = \frac{d}{r} = \frac{15}{8 + s}.$$

When I canoed upstream, I traveled 15 miles (so $d = 15$) and the rate that I traveled was $r = 8 - s$ mph (the difference of my speed in still water and the rate of the current). So the time spent traveling upstream (t_2) was

$$t_2 = \frac{d}{r} = \frac{15}{8 - s}.$$

Thus, if we add the time spent traveling downstream and the time spent traveling upstream, we should get 4 hours:

$$4 = t_1 + t_2$$

$$\Rightarrow \; 4 = \frac{15}{8+s} + \frac{15}{8-s}$$

We need to solve this equation for s to answer the question:

$$
\begin{aligned}
& 4 = \frac{15}{8+s} + \frac{15}{8-s} \\
\Rightarrow \quad & (8+s)(8-s)\cdot 4 = \left(\frac{15}{8+s} + \frac{15}{8-s}\right)\cdot(8+s)(8-s) \\
\Rightarrow \quad & (64 + 8s - 8s - s^2)\cdot 4 = 15\cdot(8-s) + 15\cdot(8+s) \\
\Rightarrow \quad & (64 - s^2)\cdot 4 = 120 - 15s + 120 + 15s \\
\Rightarrow \quad & 256 - 4s^2 = 240 \\
\Rightarrow \quad & 16 = 4s^2 \\
\Rightarrow \quad & 4 = s^2 \\
\Rightarrow \quad & s = 2 \quad \text{or} \quad s = -2
\end{aligned}
$$

The way that we defined s implies that only positive results are sensible, so we need to throw out the solution –2. Let's check if $s = 2$ is a solution:

If $s = 2$,

$$
\begin{aligned}
4 = \frac{15}{8+s} + \frac{15}{8-s} \quad & \Rightarrow \quad 4 \; ? \; \frac{15}{8+2} + \frac{15}{8-2} \\
& \Rightarrow \quad 4 \; ? \; \frac{15}{10} + \frac{15}{6} \\
& \Rightarrow \quad 4 \; ? \; \frac{3}{2} + \frac{5}{2} \\
& \Rightarrow \quad 4 = 4
\end{aligned}
$$

So 2 checks as a solution.

We can conclude that the current of the Winding River is 2 mph.

EXAMPLE: On Tuesday and Wednesday of last week I left home for school at the same time. On Tuesday I drove 40 mph (miles per hour) on my way to school and arrived 3 minutes late. On Wednesday I drove 50 mph on my way to school and arrived 2 minutes early. How far do I live from work?

SOLUTION:

To solve this problem let's set the time spent traveling to school on Tuesday equal to the time spent traveling to school on Wednesday BUT since it takes longer on Tuesday than Wednesday, we'll have to subtract a few minutes from Tuesday's time and add a couple of minutes to Wednesday's time in order for the two times to be equal. But before we can do anything, we need to **define a variable** that represents the quantity that we are looking for:

Let d represent the distance (in miles) between my house and school.

Using the formula $t = \frac{d}{r}$, we see that on Tuesday (when I drove 40 mph) it took me $\frac{d}{40}$ hours to drive to school and on Wednesday (when I drove 50 mph) it took me $\frac{d}{50}$ hours.

In order to set these times equal, we need to adjust for how many minutes I was early or late. Since I was three minutes (i.e., $\frac{3}{60}$ of an hour) late on Tuesday and two minutes (i.e., $\frac{2}{60}$ of an hour) early on Thursday we obtain the following equation:

$$\frac{d}{40} - \frac{3}{60} = \frac{d}{50} + \frac{2}{60}$$

Let's solve this equation for d to answer the question.

$$\begin{aligned}
& \frac{d}{40} - \frac{3}{60} = \frac{d}{50} + \frac{2}{60} \\
\Rightarrow\quad & 6000 \cdot \left(\frac{d}{40} - \frac{3}{60}\right) = \left(\frac{d}{50} + \frac{2}{60}\right) \cdot 6000 \qquad \text{(multiply both sides by the LCD)} \\
\Rightarrow\quad & 150d - 300 = 120d + 200 \\
\Rightarrow\quad & 30d = 500 \\
\Rightarrow\quad & d = \tfrac{50}{3} \approx 16.67
\end{aligned}$$

(Be sure to check the solution!)

So I live about 16.67 miles from school.