Section III: Rational Expressions, Equations, and Functions

Module 5: Solving Rational Equations

EXAMPLE: Suppose that the function $p(t) = \dfrac{2(t^2 - 4)}{t^2 + 1}$ represents the daily profits (in hundreds of dollars) of a small catering company t years after January 1, 2003. When will the daily profits reach \$100?

SOLUTION:

Since we need to determine when profits are \$100 and since $p(t)$ represents daily profits (in hundreds) t years after January 1, 2003, we need to determine what t-value makes $p(t) = 1$, i.e., we need to solve the *rational equation* $\dfrac{2(t^2 - 4)}{t^2 + 1} = 1$

$$\Rightarrow \quad \frac{2(t^2 - 4)}{t^2 + 1} = 1$$

$$\Rightarrow \quad (t^2 + 1)\cdot\frac{2(t^2 - 4)}{t^2 + 1} = 1\cdot(t^2 + 1)$$

(multiply both sides of the equation the LCD to eliminate fractions)

$$\Rightarrow \quad 2(t^2 - 4) = t^2 + 1$$

$$\Rightarrow \quad 2t^2 - 8 = t^2 + 1$$

$$\Rightarrow \quad t^2 - 9 = 0$$

$$\Rightarrow \quad (t - 3)(t + 3) = 0$$

(set equal to zero and factor)

$$\Rightarrow \quad t = 3 \quad \text{or} \quad t = -3$$

Since the $p(t)$ only describes profits *after* January 1, 2003, only $t = 3$ is a possible solution. Let's check if $t = 3$ is a solution, i.e., let's check if $p(3)$ equals 1:

$$p(3) = \frac{2\left((3)^2 - 4\right)}{(3)^2 + 1}$$
$$= 1$$

Since $p(3) = 1$, we can conclude that $t = 3$ is a solution to $p(t) = 1$. Thus, daily profits will reach \$100 on January 1, 2006. (Note that to answer the question we *need* this last sentence since we were asked to determine *when* daily profits reach \$100.)

As you can see from the solution above, to solve a rational equation it is important to get rid of the fractions, and the easiest way to do this is to **multiply both sides of the equation by the LCD** (least common denominator) of all involved rational expressions.

KEY POINT: *When solving rational equations it is necessary to check your solutions*. You need to check solutions since it is possible to obtain incorrect solutions to rational equations despite doing correct mathematics! After checking, you should write your solution as a ***solution set***, i.e., a set containing all of the solutions.

EXAMPLE: Solve the equation $\dfrac{x^2-5}{x-3}=\dfrac{4}{x-3}$.

SOLUTION:

$$\frac{x^2-5}{x-3}=\frac{4}{x-3}$$

$$\Rightarrow (x-3)\cdot\frac{x^2-5}{x-3}=\frac{4}{x-3}\cdot(x-3) \qquad \text{(multiply both sides by the LCD)}$$

$$\Rightarrow \quad x^2-5=4$$

$$\Rightarrow \quad x^2-9=0$$

$$\Rightarrow (x-3)(x+3)=0$$

$$\Rightarrow \quad x=3 \quad \text{or} \quad x=-3$$

Now we need to check our solutions. We will check both algebraically and graphically.

CHECK SOLUTIONS ALGEBRAICALLY:

$x=-3$:

$$\frac{x^2-5}{x-3}=\frac{4}{x-3}$$

$$\frac{(-3)^2-5}{(-3)-3} \; ? \; \frac{4}{(-3)-3}$$

$$\frac{9-5}{-6} \; ? \; \frac{4}{-6}$$

$$-\frac{4}{6}=-\frac{4}{6}$$

$x=3$:

$$\frac{x^2-5}{x-3}=\frac{4}{x-3}$$

$$\frac{(3)^2-5}{(3)-3} \; ? \; \frac{4}{(3)-3}$$

$$\frac{9-5}{0} \; ? \; \frac{4}{0}$$

These expressions aren't defined (since their denominators are zero) so 3 is not a solution.

Thus, since –3 checks but 3 doesn't, the solution set is $\{-3\}$.

CHECK SOLUTIONS GRAPHICALLY:

To check the equation graphically, we will graph the left- and right-hand sides of the equation separately and observe where the two graphs intersect, i.e., we will graph $y_1 = \dfrac{x^2 - 5}{x - 3}$ and $y_2 = \dfrac{4}{x - 3}$ and determine the x-values at which the graphs intersect. If $x = -3$ is the only solution (as our algebraic check suggests), then the graphs of the two equations should intersect **only** where $x = -3$.

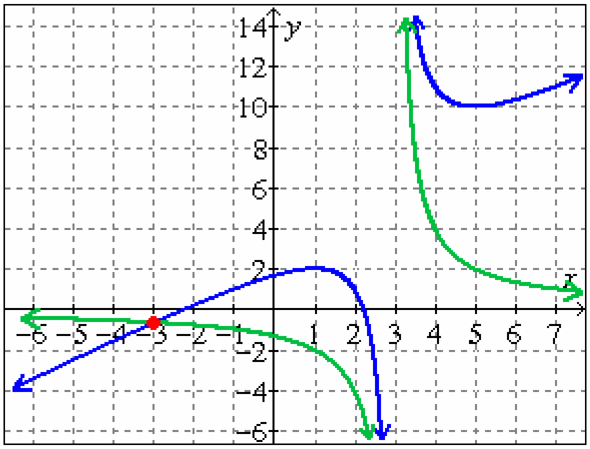

Figure 1: The graphs of $y_1 = \dfrac{x^2 - 5}{x - 3}$ and $y_2 = \dfrac{4}{x - 3}$. Note that they intersect where $x = -3$ verifying that –3 is the only solution to the given equation.

EXAMPLE: Solve the equation $\dfrac{2}{p+3} + \dfrac{1}{p^2 - 9} = \dfrac{1}{5}$.

SOLUTION:

$$\frac{2}{p+3} + \frac{1}{p^2 - 9} = \frac{1}{5}$$

$$\Rightarrow \quad \frac{2}{p+3} + \frac{1}{(p+3)(p-3)} = \frac{1}{5}$$

$$\Rightarrow 5(p+3)(p-3)\cdot\left(\frac{2}{p+3} + \frac{1}{(p+3)(p-3)}\right) = \frac{1}{5}\cdot 5(p+3)(p-3)$$

$$\Rightarrow \quad 5\cdot 2(p-3) + 5 = (p+3)(p-3)$$

$$\Rightarrow \quad 10p - 30 + 5 = p^2 - 9$$

$$\Rightarrow \quad 0 = p^2 - 10p + 16$$

$$\Rightarrow \quad 0 = (p-2)(p-8)$$

$$\Rightarrow \quad p = 2 \text{ or } p = 8$$

Now we need to check our solutions. We will check both algebraically and graphically.

CHECK SOLUTIONS ALGEBRAICALLY:

$p = 2:$

$$\frac{2}{(2)+3}+\frac{1}{(2)^2-9}=\frac{2}{5}+\frac{1}{4-9}$$
$$=\frac{2}{5}-\frac{1}{5}$$
$$=\frac{1}{5}$$

$p = 8:$

$$\frac{2}{(8)+3}+\frac{1}{(8)^2-9}=\frac{2}{11}+\frac{1}{64-9}$$
$$=\frac{2}{11}+\frac{1}{55}$$
$$=\frac{10}{55}+\frac{1}{55}$$
$$=\frac{11}{55}$$
$$=\frac{1}{5}$$

So both solutions check algebraically so the solution set for the given equation is $\{2, 8\}$

CHECK SOLUTIONS GRAPHICALLY:

In order to check if $p = 2$ and $p = 8$ are solutions, we need to see if the left- and right-hand sides of the equation $\frac{2}{p+3}+\frac{1}{p^2-9}=\frac{1}{5}$ are equal when $p = 2$ and when $p = 8$. Thus, we need to determine if the graphs of $f(p)=\frac{2}{p+3}+\frac{1}{p^2-9}$ and $y=\frac{1}{5}$ intersect when $p = 2$ and $p = 8$. (Note that $\frac{1}{5}=0.2$.)

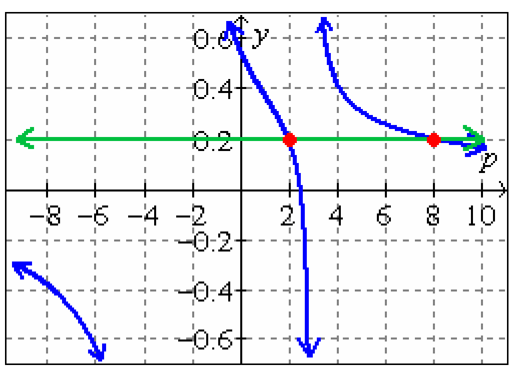

Figure 2: The graphs of $f(p)=\frac{2}{p+3}+\frac{1}{p^2-9}$ and $y=\frac{1}{5}=0.2$. Notice that they intersect when $p = 2$ and $p = 8$ verifying that both 2 and 8 solve the given equation, so the solution set for the equation $\frac{2}{p+3}+\frac{1}{p^2-9}=\frac{1}{5}$ is $\{2, 8\}$.

Try this one yourself and check your answer.

Solve the equation $\frac{2}{7} - \frac{1}{2x} = \frac{1}{4}$.

SOLUTION:

To solve this rational equation, we need to multiply both sides of the equation by the LCD. Notice that the LCD is $7 \cdot 4 \cdot x$, i.e., $28x$:

$$\frac{2}{7} - \frac{1}{2x} = \frac{1}{4}$$

$$\Rightarrow \quad 28x \cdot \left(\frac{2}{7} - \frac{1}{2x} \right) = \frac{1}{4} \cdot 28x \qquad \text{(multiply both sides of the equation by the LCD)}$$

$$\Rightarrow \quad \frac{28x \cdot 2}{7} - \frac{28x \cdot 1}{2x} = \frac{1 \cdot 28x}{4}$$

$$\Rightarrow \quad \frac{\cancel{7} \cdot 4x \cdot 2}{\cancel{7}} - \frac{\cancel{2x} \cdot 14}{\cancel{2x}} = \frac{\cancel{4} \cdot 7x}{\cancel{4}} \qquad \text{(factor the numerator to facilitate canceling)}$$

$$\Rightarrow \quad 8x - 14 = 7x$$

$$\Rightarrow \quad x = 14$$

We will check our solution *algebraically*.

$x = 14$:

$$\frac{2}{7} - \frac{1}{2(14)} = \frac{2}{7} - \frac{1}{28}$$

$$= \frac{4}{4} \cdot \frac{2}{7} - \frac{1}{28}$$

$$= \frac{8}{28} - \frac{1}{28}$$

$$= \frac{7}{28}$$

$$= \frac{1}{4}$$

Since $x = 14$ checks, we can conclude that the solution set for the equation $\frac{2}{p+3} + \frac{1}{p^2 - 9} = \frac{1}{5}$ is $\{14\}$.