Section III: Rational Expressions, Equations, and Functions

Module 4: Complex Rational Expressions

DEFINITION: A **complex rational expression** is a rational expression in which the numerator and/or denominator contains rational expressions (so that there are rational expressions inside of a rational expression).

EXAMPLES OF COMPLEX RATIONAL EXPRESSIONS:

a. $$\frac{\frac{1}{a}+\frac{1}{b}}{\frac{a}{a+b}}$$

b. $$\frac{2+\frac{2}{x}}{4-\frac{4}{x^2}}$$

c. $$\frac{\frac{3}{p-1}+\frac{1}{p-2}}{\frac{5}{p-2}+\frac{2}{p-1}}$$

We will be interested in simplifying complex rational expressions, i.e., writing as equivalent simple rational expressions. There are two standard techniques to simplify complex rational expressions:

TECHNIQUE 1	Write the numerator and denominator of the complex rational expression as single rational expression, and then convert division to multiplication.
TECHNIQUE 2	Multiply the numerator and denominator of the complex rational expression by the least common denominator (LCD) of all of the involved expressions.

You will be allowed to use the method you prefer on all exams and graded work, but you are encouraged to become familiar with both methods. In the examples below, we simplify the same expressions using both methods. (In Example 1 we use Technique 1 and in Example 2 we use Technique 2.)

EXAMPLE 1: Simplify the complex rational expressions below using **TECHNIQUE 1**.

a. $$\frac{\frac{1}{a}+\frac{1}{b}}{\frac{a}{a+b}}$$

b. $$\frac{2+\frac{2}{x}}{4-\frac{4}{x^2}}$$

c. $$\frac{\frac{3}{p-1}+\frac{1}{p-2}}{\frac{5}{p-2}+\frac{2}{p-1}}$$

SOLUTIONS:

a. $$\frac{\frac{1}{a}+\frac{1}{b}}{\frac{a}{a+b}} = \frac{\frac{1}{a}\cdot\frac{b}{b}+\frac{1}{b}\cdot\frac{a}{a}}{\frac{a}{a+b}}$$ (obtain a common denominator for the numerator)

$$= \frac{\frac{b+a}{ab}}{\frac{a}{a+b}}$$

$$= \frac{b+a}{ab}\cdot\frac{a+b}{a}$$ (convert division into multiplication)

$$= \frac{(a+b)^2}{a^2b}$$

b. $$\frac{2+\frac{2}{x}}{4-\frac{4}{x^2}} = \frac{2\cdot\frac{x}{x}+\frac{2}{x}}{4\cdot\frac{x^2}{x^2}-\frac{4}{x^2}}$$ (obtain a common denominator in both the numerator and the denominator)

$$= \frac{\frac{2x+2}{x}}{\frac{4x^2-4}{x^2}}$$

$$= \frac{2x+2}{x}\cdot\frac{x^2}{4x^2-4}$$ (convert division into multiplication)

$$\left.\begin{aligned}
&= \frac{2(x+1)}{x}\cdot\frac{x\cdot x}{4(x^2-1)} \\
&= \frac{\cancel{2}\,\cancel{(x+1)}}{\cancel{x}}\cdot\frac{\cancel{x}\cdot x}{\cancel{2}\cdot 2(x-1)\,\cancel{(x+1)}} \\
&= \frac{x}{2(x-1)}
\end{aligned}\right\}$$ (factor and simplify)

c. $$\frac{\dfrac{3}{p-1}+\dfrac{1}{p-2}}{\dfrac{5}{p-2}+\dfrac{2}{p-1}} = \frac{\dfrac{3}{p-1}\cdot\dfrac{p-2}{p-2}+\dfrac{1}{p-2}\cdot\dfrac{p-1}{p-1}}{\dfrac{5}{p-2}\cdot\dfrac{p-1}{p-1}+\dfrac{2}{p-1}\cdot\dfrac{p-2}{p-2}}$$ (obtain a common denominator in numerator and denominator)

$$= \frac{\dfrac{3(p-2)+p-1}{(p-1)(p-2)}}{\dfrac{5(p-1)+2(p-2)}{(p-1)(p-2)}}$$

$$= \frac{3(p-2)+p-1}{\cancel{(p-1)}\,\cancel{(p-2)}}\cdot\frac{\cancel{(p-1)}\,\cancel{(p-2)}}{5(p-1)+2(p-2)}$$

$$= \frac{3(p-2)+p-1}{5(p-1)+2(p-2)}$$

$$= \frac{3p-6+p-1}{5p-5+2p-4}$$

$$= \frac{4p-7}{7p-9}$$

EXAMPLE 2: Simplify the complex rational expressions below using **TECHNIQUE 2**.

a. $\dfrac{\dfrac{1}{a}+\dfrac{1}{b}}{\dfrac{a}{a+b}}$ **b.** $\dfrac{2+\dfrac{2}{x}}{4-\dfrac{4}{x^2}}$ **c.** $\dfrac{\dfrac{3}{p-1}+\dfrac{1}{p-2}}{\dfrac{5}{p-2}+\dfrac{2}{p-1}}$

SOLUTIONS:

a. $$\frac{\dfrac{1}{a}+\dfrac{1}{b}}{\dfrac{a}{a+b}} = \frac{\dfrac{1}{a}+\dfrac{1}{b}}{\dfrac{a}{a+b}}\cdot\frac{ab(a+b)}{ab(a+b)}$$ (multiply numerator and denominator by LCD)

$$= \frac{\left(\dfrac{1}{a}+\dfrac{1}{b}\right)\cdot ab(a+b)}{\left(\dfrac{a}{a+b}\right)\cdot ab(a+b)}$$

$$= \frac{b(a+b)+a\cdot(a+b)}{a^2b}$$

$$= \frac{(a+b)(b+a)}{a^2b}$$

$$= \frac{(a+b)^2}{a^2b}$$

(factor and simplify)

b. $$\frac{2+\frac{2}{x}}{4-\frac{4}{x^2}} = \frac{2+\frac{2}{x}}{4-\frac{4}{x^2}}\cdot\frac{x^2}{x^2}$$ (multiply numerator and denominator by LCD)

$$= \frac{\left(2+\frac{2}{x}\right)x^2}{\left(4-\frac{4}{x^2}\right)x^2}$$

$$= \frac{2x^2+2x}{4x^2-4}$$

$$= \frac{2x(x+1)}{4(x^2-1)}$$

$$= \frac{\cancel{2}x\,\cancel{(x+1)}}{\cancel{2}\cdot 2(x-1)\,\cancel{(x+1)}}$$ (factor and simplify)

$$= \frac{x}{2(x-1)}$$

c. $$\frac{\frac{3}{p-1}+\frac{1}{p-2}}{\frac{5}{p-2}+\frac{2}{p-1}} = \frac{\frac{3}{p-1}+\frac{1}{p-2}}{\frac{5}{p-2}+\frac{2}{p-1}}\cdot\frac{(p-1)(p-2)}{(p-1)(p-2)}$$ (Multiply numerator and denominator by LCD)

$$= \frac{\left(\frac{3}{p-1}+\frac{1}{p-2}\right)\cdot(p-1)(p-2)}{\left(\frac{5}{p-2}+\frac{2}{p-1}\right)\cdot(p-1)(p-2)}$$

$$= \frac{3(p-2)+p-1}{5(p-1)+2(p-2)}$$

$$= \frac{3p-6+p-1}{5p-5+2p-4}$$

$$= \frac{4p-7}{7p-9}$$

EXAMPLE: If $f(x) = \dfrac{x+2}{x-5}$ find and simplify $f\left(\dfrac{2}{a}\right)$.

SOLUTION:

$$f\left(\frac{2}{a}\right) = \frac{\frac{2}{a}+2}{\frac{2}{a}-5}\cdot\frac{a}{a} \qquad \text{(multiply numerator and denominator by LCD)}$$

$$= \frac{\left(\frac{2}{a}+2\right)a}{\left(\frac{2}{a}-5\right)a}$$

$$= \frac{2+2a}{2-5a}$$

Try this one yourself and check your answer.

If $g(x) = \dfrac{1}{x}$, find and simplify the expression $\dfrac{g(x+h)-g(x)}{h}$.

SOLUTION:

$$\frac{g(x+h)-g(x)}{h} = \frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

$$= \frac{\frac{1}{x+h}-\frac{1}{x}}{h}\cdot\frac{x(x+h)}{x(x+h)} \qquad \text{(multiply numerator and denominator by LCD)}$$

$$= \frac{\left(\frac{1}{x+h}-\frac{1}{x}\right)\cdot x(x+h)}{h\cdot x(x+h)}$$

$$= \frac{x-(x+h)}{h\cdot x(x+h)}$$

$$= \frac{-\cancel{h}}{\cancel{h}\cdot x(x+h)}$$

$$= \frac{-1}{x(x+h)}$$