Section III: Rational Expressions, Equations, and Functions

Module 1: Introduction to Rational Functions

DEFINITION: A **rational function** is a ratio of polynomial functions. If p and q are polynomial functions, then $r(x) = \dfrac{p(x)}{q(x)}$ is a rational function. Since the denominator of a fraction can never equal zero, the domain of r is the set $\{x \mid x \in \mathbb{R} \text{ and } q(x) \neq 0\}$.

EXAMPLES OF RATIONAL FUNCTIONS:

a. $f(x) = \dfrac{x^2 + 4x - 13}{x - 3}$

The domain of f is $\{x \mid x \in \mathbb{R} \text{ and } x \neq 3\}$ since when $x = 3$ the denominator is zero.

b. $g(x) = \dfrac{7x^5 - x^4 + 2x^2 + 5x - 6}{x^2 + 5x + 6}$

To determine the domain of g we must find the values of x which make the denominator zero. So we need to solve $x^2 + 5x + 6 = 0$.

$$\begin{aligned} & x^2 + 5x + 6 = 0 \\ \Rightarrow\ & (x + 2)(x + 3) = 0 \\ \Rightarrow\ & x + 2 = 0 \quad \text{or} \quad x + 3 = 0 \\ \Rightarrow\ & x = -2 \quad \text{or} \quad x = -3 \end{aligned}$$

Thus, the domain of g is $\{x \mid x \in \mathbb{R} \text{ and } x \neq -2 \text{ and } x \neq -3\}$.

c. $h(x) = \dfrac{x}{x^2 + 7}$

The domain of h is $\mathbb{R}$ (all real numbers) since the denominator can never equal zero.

d. $k(x) = \dfrac{12}{x}$

The domain of k is $\{x \mid x \in \mathbb{R} \text{ and } x \neq 0\}$.

GRAPHS OF RATIONAL FUNCTIONS

Graphing rational functions is discussed in detail in College Algebra (MTH 111b/c). Here we only wish to get an idea about what happens to the graph of a rational function as the x-values get closer and closer to numbers that make the denominator of the function zero.

EXAMPLE: What happens to the graph of the function $f(x) = \dfrac{4}{x-2}$ as x gets closer and closer to 2 (in symbols: "$x \to 2$").

SOLUTION:

Since 2 is not in the domain of $f(x) = \dfrac{4}{x-2}$ (the domain of f is the set $\{x \,|\, x \in \mathbb{R} \text{ and } x \neq 2\}$), we cannot evaluate the function at $x = 2$. So all we can do is get an idea about what happens to the graph of f as x gets closer and closer to 2 (i.e., $x \to 2$). We can use a table of function values to see what happens to the outputs as the inputs get closer and closer to $x = 2$.

Table 1

x	$f(x) = \dfrac{4}{x-2}$
1.9	–40
1.99	–400
1.999	–4000
1.9999	–40000

Table 2

x	$f(x) = \dfrac{4}{x-2}$
2.1	40
2.01	400
2.001	4000
2.0001	40000

The values in these tables suggest that if you start from a number less than 2 and get closer and closer to 2 (e.g., 1.9, 1.99, 1.999, …) then the outputs get smaller and smaller, while if you start from a number larger than 2 and get closer and closer to 2 (e.g., 2.1, 2.01, 2.001, …) then the outputs get larger and larger. To describe this behavior people sometimes say things like, "as x approaches 2 from less than 2, the outputs approach negative infinity while as x approaches 2 from larger than 2 the outputs approach positive infinity." We can describe this behavior more technically as follows:

> The function values in Table 1 suggest that as x gets closer and closer to 2 (but remains less than 2) the outputs get smaller and smaller. To describe this behavior mathematicians usually say, "As x approaches 2 from below, $f(x)$ decreases without bound." We can write this using symbols: "As $x \to 2^-$, $f(x) \to -\infty$."
>
> The function values in Table 2 suggest that as x gets closer and closer to 2 (but remains greater than 2) the outputs get larger and larger. To describe this behavior mathematicians usually say, "As x approaches 2 from above, $f(x)$ increases without bound." We can write this using symbols: "As $x \to 2^+$, $f(x) \to \infty$."

In Figure 1, a graph of $f(x) = \dfrac{4}{x-2}$ is given. The behavior of the graph supports the analysis given above. The line $x = 2$ is called a **vertical asymptote**. The graph never crosses the vertical asymptote, which is what we would expect since the function isn't defined when $x = 2$!

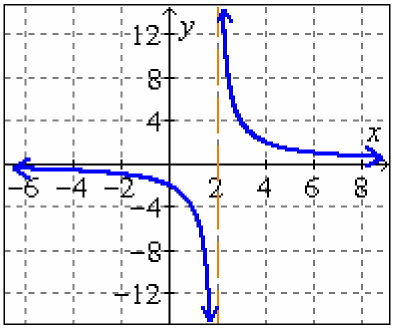

Figure 1: Graph of $f(x) = \dfrac{4}{x-2}$.

EXAMPLE: Consider the function $g(x) = \dfrac{x+4}{x^2 - x - 6}$. To determine the domain of g we need to find out which x-values make the denominator of g zero.

$$\begin{aligned} & x^2 - x - 6 = 0 \\ \Rightarrow\ & (x-3)(x+2) = 0 \\ \Rightarrow\ & x - 3 = 0 \quad \text{or} \quad x + 2 = 0 \\ \Rightarrow\ & x = 3 \quad \text{or} \quad x = -2 \end{aligned}$$

Since 3 and –2 make the denominator of g zero, these values must be excluded from the domain. Thus, the domain of g is the set $\{x \mid x \in \mathbb{R} \text{ and } x \neq 3 \text{ and } x \neq -2\}$. The graph of g has two vertical asymptotes: $x = 3$ and $x = -2$ (see Figure 2).

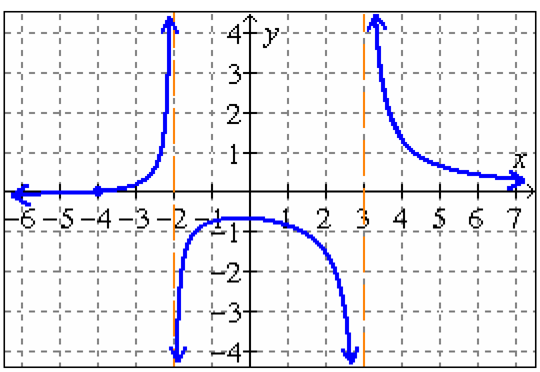

Figure 2: Graph of $g(x) = \dfrac{x+4}{x^2 - x - 6}$.

EVALUATING RATIONAL FUNCTIONS

EXAMPLE: If $f(x) = \dfrac{x^2 + 4x - 13}{x - 3}$, evaluate the following.

a. $f(2)$ **b.** $f(0)$

c. $f(3)$ **d.** $f(-4)$

SOLUTIONS:

a. $$f(2) = \frac{(2)^2 + 4(2) - 13}{(2) - 3} = \frac{4 + 8 - 13}{-1} = \frac{-1}{-1} = 1$$

b. $$f(0) = \frac{(0)^2 + 4(0) - 13}{(0) - 3} = \frac{0 + 0 - 13}{0 - 3} = \frac{-13}{-3} = \frac{13}{3}$$

c. Since 3 isn't in the domain of f (i.e., when $x = 3$ the denominator is zero) we say, "$f(3)$ is undefined."

d. $$f(-4) = \frac{(-4)^2 + 4(-4) - 13}{(-4) - 3} = \frac{16 - 16 - 13}{-7} = \frac{-13}{-7} = \frac{13}{7}$$

Try these yourself and check your answers.

If $g(x) = \dfrac{7x^5 - x^4 + 2x^2 + 5x - 6}{x^2 + 5x + 6}$, evaluate the following:

a. $g(-1)$ **b.** $g(0)$

c. $g(1)$ **d.** $g(-2)$

SOLUTIONS:

a. $$\begin{aligned} g(-1) &= \frac{7(-1)^5 - (-1)^4 + 2(-1)^2 + 5(-1) - 6}{(-1)^2 + 5(-1) + 6} \\ &= \frac{-7 - 1 + 2 - 5 - 6}{1 - 5 + 6} \\ &= \frac{-17}{2} \\ &= -\frac{17}{2} \end{aligned}$$

b. $$\begin{aligned} g(0) &= \frac{7(0)^5 - (0)^4 + 2(0)^2 + 5(0) - 6}{(0)^2 + 5(0) + 6} \\ &= \frac{0 - 0 + 0 + 0 - 6}{0 + 0 + 6} \\ &= \frac{-6}{6} \\ &= -1 \end{aligned}$$

c. $$\begin{aligned} g(1) &= \frac{7(1)^5 - (1)^4 + 2(1)^2 + 5(1) - 6}{(1)^2 + 5(1) + 6} \\ &= \frac{7 - 1 + 2 + 5 - 6}{1 + 5 + 6} \\ &= \frac{7}{12} \end{aligned}$$

d. Since the denominator of g is zero when $x = -2$, $g(-2)$ is undefined.