Solutions to "Some Additional Practice for the Midterm Exam**"**

1. a. Convert 24° into radians.

$$\begin{aligned} 24^\circ &= 24^\circ \cdot \frac{2\pi \text{ rad.}}{360^\circ} \\ &= \frac{2\pi}{15} \text{ rad.} \end{aligned}$$

b. Convert $\frac{3}{2}$ radians into degrees.

$$\begin{aligned} \frac{3}{2} \text{ rad.} &= \frac{3}{2} \text{ rad.} \cdot \frac{360^\circ}{2\pi \text{ rad.}} \\ &= \frac{540^\circ}{2\pi} \\ &= \frac{270^\circ}{\pi} \end{aligned}$$

2. Find the arc-length spanned by an angle measuring 24° on a circle of radius 30 feet.

In part **a** of #1 we found that $24^\circ = \frac{2\pi}{15}$ rad. Thus,

$$\begin{aligned} s &= r\theta \\ &= 30 \cdot \frac{2\pi}{15} \\ &= 4\pi \end{aligned}$$

and we see that the arc-length spanned by an angle measuring 24° on a circle of radius 30 feet is 4π feet.

3. Evaluate the following expressions:

a. $\sin(225^\circ) = -\sin(45^\circ)$ (since 45° is the reference angle for 225° and sine is negative in the 3rd quadrant)

$$= -\frac{\sqrt{2}}{2}$$

b. $\sin\left(\frac{5\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right)$ (since $\frac{\pi}{3}$ is the reference angle for $\frac{5\pi}{3}$ and sine is negative in the 4^{th} quadrant)

$= -\frac{\sqrt{3}}{2}$

c. $\cos(300^\circ) = \cos(60^\circ)$ (since 60° is the reference angle for 300°)

$= \frac{1}{2}$

d. $\cos\left(\frac{17\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right)$ (since $\frac{5\pi}{6}$ is coterminal with $\frac{17\pi}{6}$)

$= -\cos\left(\frac{\pi}{6}\right)$ (since $\frac{\pi}{6}$ is the reference angle for $\frac{5\pi}{6}$ and cosine is negative in the 2^{nd} quadrant)

$= -\frac{\sqrt{3}}{2}$

e. $\tan\left(\frac{3\pi}{4}\right) = \frac{\sin\left(\frac{3\pi}{4}\right)}{\cos\left(\frac{3\pi}{4}\right)}$

$= \frac{\sin\left(\frac{\pi}{4}\right)}{-\cos\left(\frac{\pi}{4}\right)}$ (since $\frac{\pi}{4}$ is the reference angle for $\frac{3\pi}{4}$ and cosine is negative in the 2^{nd} quadrant)

$= \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}}$

$= -1$

f. $\sec\left(3600^\circ\right) = \frac{1}{\cos\left(3600^\circ\right)}$

$= \frac{1}{\cos\left(360^\circ\right)}$ (since 360° is coterminal with 3600°)

$= \frac{1}{1}$

$= 1$

4. If $\sin(\theta) = \frac{\sqrt{5}}{4}$ and θ is in the second quadrant, find the exact value of the following expressions.

a. $\cos(\theta)$

We can use the Pythagorean identity to find $\cos(\theta)$:

$$\begin{aligned} & \sin^2(\theta) + \cos^2(\theta) = 1 \\ \Rightarrow\ & \left(\tfrac{\sqrt{5}}{4}\right)^2 + \cos^2(\theta) = 1 \\ \Rightarrow\ & \cos^2(\theta) = 1 - \tfrac{5}{16} \\ \Rightarrow\ & \cos^2(\theta) = \tfrac{11}{16} \\ \Rightarrow\ & \cos(\theta) = -\tfrac{\sqrt{11}}{4} \end{aligned}$$

(note that we take the negative square root since cosine is negative in the second quadrant)

b. $\cos\left(\theta - \frac{\pi}{2}\right) = \sin(\theta)$ (since the graph of $y = \cos(\theta)$ shifted right $\frac{\pi}{2}$ units becomes the graph of $y = \sin(\theta)$)

$$= \frac{\sqrt{5}}{4}$$

c. $\sin(\theta + 2\pi) = \sin(\theta)$ (since the period of $y = \sin(\theta)$ is 2π)

$$= \frac{\sqrt{5}}{4}$$

d.

$$\begin{aligned} \tan(\theta) &= \frac{\sin(\theta)}{\cos(\theta)} \\ &= \frac{\sqrt{5}/4}{-\sqrt{11}/4} \\ &= -\frac{\sqrt{5}}{\sqrt{11}} \end{aligned}$$

e.

$$\begin{aligned} \sec(\theta) &= \frac{1}{\cos(\theta)} \\ &= \frac{1}{-\sqrt{11}/4} \\ &= -\frac{4}{\sqrt{11}} \end{aligned}$$

f. $\csc(\theta) = \frac{1}{\sin(\theta)}$

$= \frac{1}{\sqrt{5}/4}$

$= \frac{4}{\sqrt{5}}$

5. Find all of the solutions to the following equations on the interval $[0, 2\pi)$:

a. $2\cos(\theta) = 1$

$$\begin{aligned} & 2\cos(\theta) = 1 \\ \Rightarrow \quad & \cos(\theta) = \tfrac{1}{2} \end{aligned}$$

In the first quadrant, we know that $\theta = \frac{\pi}{3}$ is a solution.

In the second and third quadrant cosine is negative, so there are no solutions.

In the fourth quadrant we know that $\theta = \frac{5\pi}{3}$ is a solution.

Thus the solution set on the interval $[0, 2\pi)$ is $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$.

b. $\sin(2\theta) + 3 = 4$

$$\begin{aligned} & \sin(2\theta) + 3 = 4 \\ \Rightarrow \quad & \sin(2\theta) = 1 \end{aligned}$$

Since $\sin(x) = 1$ when $x = \frac{\pi}{2} + 2k\pi$ (where $k \in \mathbb{Z}$), we need:

$$2\theta = \frac{\pi}{2} + 2k\pi$$

$$\Rightarrow \theta = \frac{\frac{\pi}{2} + 2k\pi}{2}$$

$$\Rightarrow \theta = \frac{\pi}{4} + k\pi$$

Since $0 \le \theta < 2\pi$, we can only take $k = 0$ and $k = 1$ in the statement above. Thus, the solution set is $\left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$.

6. Find *all* of the solutions to the following equations:

a. $3\sin(x) + 4 = 5$

$$\begin{aligned} & 3\sin(x) + 4 = 5 \\ \Rightarrow \quad & 3\sin(x) = 1 \\ \Rightarrow \quad & \sin(x) = \tfrac{1}{3} \end{aligned}$$

Since this sine value isn't one we are familiar with, we will need to use the inverse sine function (i.e., arcsine function).

$$\begin{aligned} & \sin(x) = \tfrac{1}{3} \\ \Rightarrow \quad & x = \sin^{-1}\left(\tfrac{1}{3}\right) + 2k\pi, \\ & \text{or } \; x = \pi - \sin^{-1}\left(\tfrac{1}{3}\right) + 2k\pi \quad \text{where } k \in \mathbb{Z} \end{aligned}$$

b. $7 + 3\sqrt{2}\cos(4t) = 4$

$$\begin{aligned} & 7 + 3\sqrt{2}\cos(4t) = 4 \\ \Rightarrow \quad & 3\sqrt{2}\cos(4t) = -3 \\ \Rightarrow \quad & \cos(4t) = \tfrac{-3}{3\sqrt{2}} \\ \Rightarrow \quad & \cos(4t) = -\tfrac{1}{\sqrt{2}} \end{aligned}$$

$$\Rightarrow \quad \begin{cases} 4t = \frac{3\pi}{4} + 2k\pi, \quad k \in \mathbb{Z} \\ \text{OR} \\ 4t = -\frac{3\pi}{4} + 2k\pi, \quad k \in \mathbb{Z} \end{cases}$$

$$\Rightarrow \quad \begin{cases} t = \frac{3\pi}{16} + \frac{k\pi}{2}, \quad k \in \mathbb{Z} \\ \text{OR} \\ t = -\frac{3\pi}{16} + \frac{k\pi}{2}, \quad k \in \mathbb{Z} \end{cases}$$

7. Use the sine and cosine functions to find the coordinates of the point P in Figure 1.

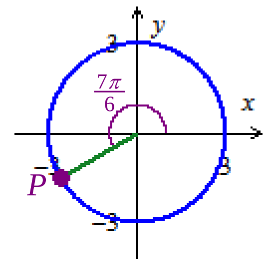


Figure 1

Coordinates of P:

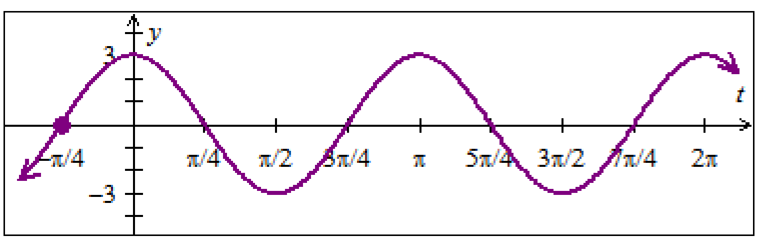
$$\left(3\cos\left(\frac{7\pi}{6}\right),\ 3\sin\left(\frac{7\pi}{6}\right)\right) = \left(3\cdot\left(-\frac{\sqrt{3}}{2}\right),\ 3\cdot\left(-\frac{1}{2}\right)\right)$$
$$= \left(-\frac{3\sqrt{3}}{2},\ -\frac{3}{2}\right)$$

8. Sketch a graph of the function $g(t) = 3\sin\left(2t + \frac{\pi}{2}\right)$. State the period, midline, and amplitude of g.

First, let's write the function in "standard form":

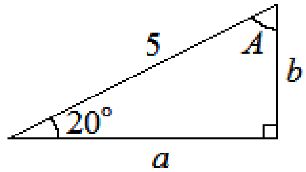
$$g(t) = 3\sin\left(2t + \frac{\pi}{2}\right)$$
$$= 3\sin\left(2\left(t + \frac{\pi}{4}\right)\right)$$

Thus, this is a function of the form $g(t) = A\sin\left(\omega(t-h)\right) + k$ where $A = 3$, $\omega = 2$, $h = -\frac{\pi}{4}$, and $k = 0$. Since $A = 3$ the function has **amplitude 3 units**. Using the fact that $\omega = 2$, we can find the period: period $= \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$. So the **period is π units**. Since $k = 0$, the **midline is $y = 0$**. Since $h = -\frac{\pi}{4}$, we need to "start" our sine wave at $t = -\frac{\pi}{4}$, i.e., shift the wave left $\frac{\pi}{4}$ units. Below is the graph of $g(t) = 3\sin\left(2\left(t + \frac{\pi}{4}\right)\right)$.



A graph of $g(t) = 3\sin\left(2\left(t + \frac{\pi}{4}\right)\right)$.

9. Find the missing sides a and b and the missing angle A for the right-triangle below. (The triangle may not be drawn to scale.)



We can find A easily, using the fact that the sum of the angles in a triangle is 180°:

$$A + 20^\circ + 90^\circ = 180^\circ$$
$$\Rightarrow \quad A = 180^\circ - 20^\circ - 90^\circ$$
$$\Rightarrow \quad A = 70^\circ$$

We can find b using sine:

$$\sin(20^\circ) = \frac{b}{5}$$
$$\Rightarrow \quad b = 5\sin(20^\circ)$$
$$\approx 1.71$$

(you won't be expected to obtain approximations on the exam since you won't have a calculator)

and we can use cosine to find a:

$$\cos(20^\circ) = \frac{a}{5}$$
$$\Rightarrow \quad a = 5\cos(20^\circ)$$
$$\approx 4.698$$

(you won't be expected to obtain approximations on the exam since you won't have a calculator)

10. a. Find possible algebraic rule for the function $y = f(t)$ graphed in Figure 2.

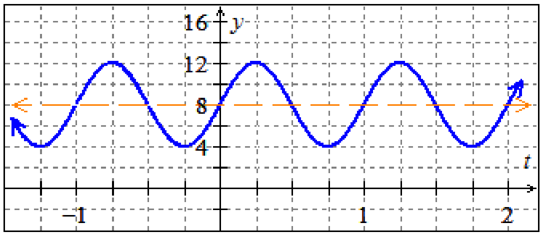


Figure 2: Graph of $y = f(t)$.

Let's write a rule involving sine, so our rule will have the form $p(x) = A\sin\left(\omega(x-h)\right) + k$ and we need to determine the values of A, ω, h, and k.

- The midline is the line midway between the function's maximum and minimum output values. The function's maximum output value is 12 and its minimum output value is 4. Since 8 is the average of these values, the midline is $y = 8$ so $k = 8$.
- The amplitude is the distance between the function's maximum output value, 12, and its midline $y = 8$, which is 4 units. Therefore, $|A| = 4$.
- The function completes one period between $x = 0$ and $x = 1$. Thus, the period of the function is $1 - 0 = 1$. To find ω we need to solve $1 = 2\pi \cdot \frac{1}{\omega}$:

$$1 = 2\pi \cdot \frac{1}{\omega}$$
$$\Rightarrow \quad \omega = 2\pi$$

- Near the y-axis, the graph of $y = \sin(x)$ is increasing and passes through its midline, so we need to look for a spot in the graph of $y = p(x)$ where it shows this behavior, and one such spot is on the y-axis, at $x = 0$ so we don't need a horizontal shift.

Therefore, an algebraic rule for the graphed function is $f(t) = 4\sin(2\pi t) + 8$.

(There are MANY other answers.)

b. Use your answer to part **a** to find exact solutions to $f(t) = 10$

$$\begin{aligned}
& f(t) = 10 \\
\Rightarrow\quad & 4\sin(2\pi t) + 8 = 10 \\
\Rightarrow\quad & 4\sin(2\pi t) = 2 \\
\Rightarrow\quad & \sin(2\pi t) = \tfrac{1}{2} \\
\Rightarrow\quad & 2\pi t = \tfrac{\pi}{6} + 2k\pi \quad \text{or} \quad 2\pi t = \tfrac{5\pi}{6} + 2k\pi, \quad k \in \mathbb{Z} \\
\Rightarrow\quad & t = \frac{\frac{\pi}{6} + 2k\pi}{2\pi} \quad \text{or} \quad t = \frac{\frac{5\pi}{6} + 2k\pi}{2\pi}, \quad k \in \mathbb{Z} \\
\Rightarrow\quad & t = \tfrac{1}{12} + k \quad \text{or} \quad t = \tfrac{5}{12} + k, \quad k \in \mathbb{Z}
\end{aligned}$$

11. Evaluate the following:

a. $\sin\left(\cos^{-1}\left(-\tfrac{1}{2}\right)\right)$

$$\sin\left(\cos^{-1}\left(-\tfrac{1}{2}\right)\right) = \sin\left(\tfrac{2\pi}{3}\right) \qquad \text{(since the range of } y = \cos^{-1}(x) \text{ is } [0, \pi]\text{)}$$

$$= \tfrac{\sqrt{3}}{2}$$

b. $\sin^{-1}\left(\sin\left(\tfrac{7\pi}{4}\right)\right)$

$$\sin^{-1}\left(\sin\left(\tfrac{7\pi}{4}\right)\right) = \sin^{-1}\left(-\tfrac{\sqrt{2}}{2}\right)$$

$$= -\tfrac{\pi}{4} \qquad \text{(since the range of } y = \sin^{-1}(x) \text{ is } \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]\text{)}$$

c. $\cos^{-1}\left(\sin\left(\tfrac{4\pi}{3}\right)\right)$

$$\cos^{-1}\left(\sin\left(\tfrac{4\pi}{3}\right)\right) = \cos^{-1}\left(-\tfrac{\sqrt{3}}{2}\right)$$

$$= \tfrac{5\pi}{6} \qquad \text{(since the range of } y = \cos^{-1}(x) \text{ is } [0, \pi]\text{)}$$