Solutions to "Midterm Exam"

1. Circle **T** for *true* or **F** for *false*. (You don't need to justify your answers.)

a. **F** $\cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right) = \frac{7\pi}{6}$.

(since $\frac{7\pi}{6}$ isn't in the range of the inverse cosine function)

b. **T** $\cos(-t) = \cos(t)$ for all t.

(since cosine is an EVEN function, i.e., it's symmetric about the y-axis)

c. **T** $\sin\left(t + \frac{\pi}{2}\right) = \cos(t)$ for all t.

(since shifting $y = \sin(t)$ left $\frac{\pi}{2}$ units does produce $y = \cos(t)$)

d. **T** An angle of measure 2 radians is larger than an angle of measure 2°.

(since $2\text{ rad} \approx 114.6^\circ > 1^\circ$)

e. **F** An angle of measure 45° in a circle of radius 1 unit is smaller than an angle of measure 45° in a circle of radius 2 units.

(since $45^\circ = 45^\circ$ no matter what circle it's in)

2. Find the length of the arc spanned by an angle of 200° in a circle of radius 15 feet. [Provide a completely simplified, exact numerical value.]

To compute the arc-length we can use the formula $s = r \cdot \theta$, but we need θ in radians to use this formula:

$$\begin{aligned}\theta = 200^\circ &= 200^\circ \cdot \frac{\pi}{180^\circ}\\ &= \frac{200\pi}{180}\\ &= \frac{10\pi}{9}\end{aligned}$$

Now we can calculate the arc-length:

$$\begin{aligned}s = r \cdot \theta &= 15 \cdot \frac{10\pi}{9}\\ &= \frac{150\pi}{9}\\ &= \frac{3 \cdot 50\pi}{3 \cdot 3}\\ &= \frac{50\pi}{3}\end{aligned}$$

Thus, the desired arc-length is $\frac{50\pi}{3}$ feet.

3. Find the ***exact*** value for each of the following expressions. Be sure to use proper notation to communicate your answer, i.e., link the given expression and your answer with an equal sign. If the given expression is undefined write, "*The expression is undefined*." An example has been provided.

ex. $\sin(0)$

$$\sin(0) = 0$$

(Write all of this to communicate what "$\sin(0)$" *equals*.)

a. $\sin\left(\frac{\pi}{4}\right)$.

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

b. $\cos(\pi)$.

$$\cos(\pi) = -1$$

c. $\cos\left(\frac{5\pi}{6}\right)$.

$$\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

d. $\cos\left(\frac{4\pi}{3}\right)$.

$$\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$$

e. $\sin\left(-\frac{2\pi}{3}\right)$.

$$\sin\left(-\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

f. $\sin\left(\frac{13\pi}{4}\right)$.

$$\sin\left(\frac{13\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

g. $\tan\left(\frac{11\pi}{6}\right)$.

$$\tan\left(\frac{11\pi}{6}\right) = \frac{\sin\left(\frac{11\pi}{6}\right)}{\cos\left(\frac{11\pi}{6}\right)}$$

$$= -\frac{1}{\sqrt{3}}$$

h. $\tan\left(\frac{3\pi}{2}\right)$.

$$\tan\left(\frac{3\pi}{2}\right) = \frac{\sin\left(\frac{3\pi}{2}\right)}{\cos\left(\frac{3\pi}{2}\right)}.$$

Since $\cos\left(\frac{3\pi}{2}\right) = 0$, the expression $\tan\left(\frac{3\pi}{2}\right)$ is undefined.

i. $\sec\left(\frac{\pi}{6}\right)$.

$$\sec\left(\frac{\pi}{6}\right) = \frac{1}{\cos\left(\frac{\pi}{6}\right)}$$

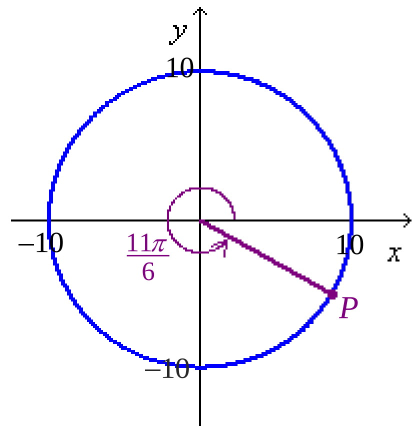
$$= \frac{2}{\sqrt{3}}$$

j. $\csc\left(\frac{8\pi}{3}\right)$.

$$\csc\left(\frac{8\pi}{3}\right) = \frac{1}{\sin\left(\frac{8\pi}{3}\right)}$$

$$= \frac{2}{\sqrt{3}}$$

4. Use the sine and cosine functions to find the exact coordinates of point P specified by the angle $\frac{11\pi}{6}$ on the circumference of a circle of radius 10 units. Be sure to ***show your use of sine and cosine*** and to provide completely simplified, exact numerical values.



Coordinates of point P:

$$\left(10\cos\left(\tfrac{11\pi}{6}\right),\, 10\sin\left(\tfrac{11\pi}{6}\right)\right) = \left(10\cdot\left(\tfrac{\sqrt{3}}{2}\right),\, 10\cdot\left(-\tfrac{1}{2}\right)\right)$$
$$= \left(5\sqrt{3},\ -5\right)$$

5. If $\cos(\theta) = \frac{\sqrt{10}}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$, find exact numerical values for the expressions given below. [Be sure to compose conclusions that *directly* communicate the values of the given expressions. See the example given in #3.]

a. $\sin(\theta)$

We can use the Pythagorean identity to find $\sin(\theta)$:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$
$$\Rightarrow \sin^2(\theta) + \left(\frac{\sqrt{10}}{5}\right)^2 = 1$$
$$\Rightarrow \sin^2(\theta) = 1 - \frac{10}{25}$$
$$\Rightarrow \sin^2(\theta) = \frac{15}{25}$$
$$\Rightarrow \sin(\theta) = -\frac{\sqrt{15}}{5}$$

(note that we take the negative square root since $\sin(\theta)$ is negative for $\frac{3\pi}{2} < \theta < 2\pi$)

b. $\tan(\theta)$.

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$
$$= \frac{-\sqrt{15}/5}{\sqrt{10}/5}$$
$$= -\frac{\sqrt{15}}{\sqrt{10}} = -\sqrt{\frac{15}{10}} = -\sqrt{\frac{3}{2}} = -\frac{\sqrt{6}}{2}$$

c. $\sec(\theta)$.

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$
$$= \frac{1}{\sqrt{10}/5}$$
$$= \frac{5}{\sqrt{10}} = \frac{5\sqrt{10}}{10} = \frac{\sqrt{10}}{2}$$

6. Determine the period, midline, and amplitude of the function $g(t) = 3\cos\left(\pi t + \frac{\pi}{2}\right) + 2$ and then use that information to draw a graph of **at least two periods** of the function.

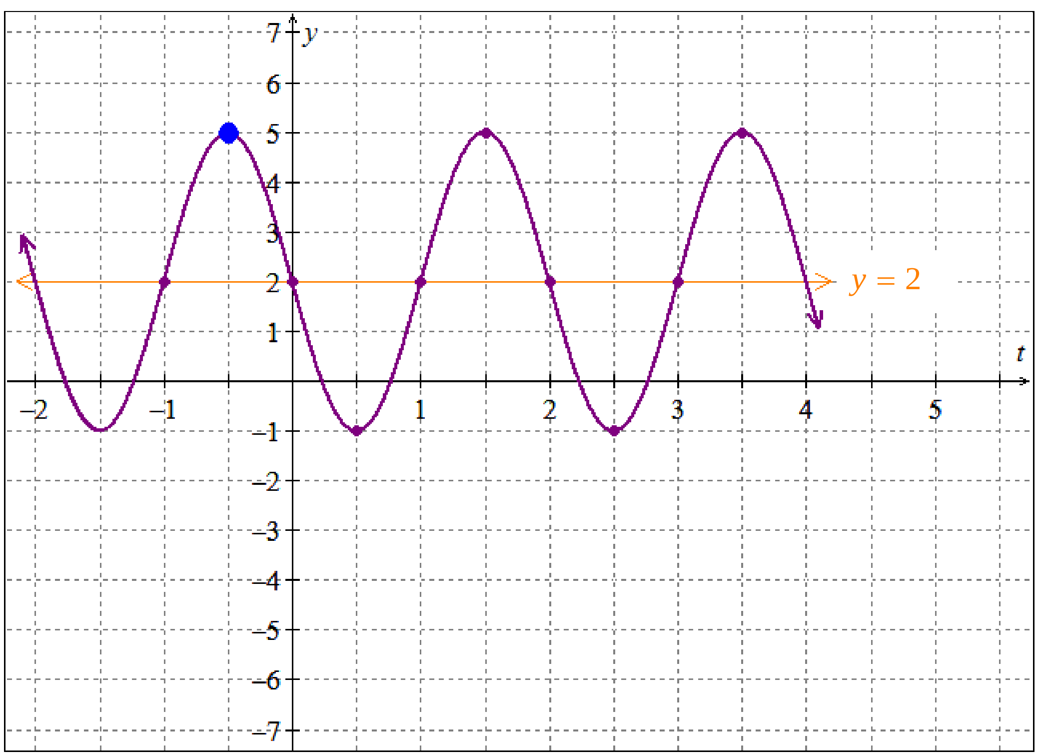
BE SURE TO LIST THE *PERIOD*, *MIDLINE*, AND *AMPLITUDE* OF THE FUNCTION AND TO LABEL THE SCALE ON THE AXES OF THE GRAPH!

Since

$$g(t) = 3\cos\left(\pi t + \tfrac{\pi}{2}\right) + 2$$
$$= 3\cos\left(\pi\left(t + \tfrac{1}{2}\right)\right) + 2,$$

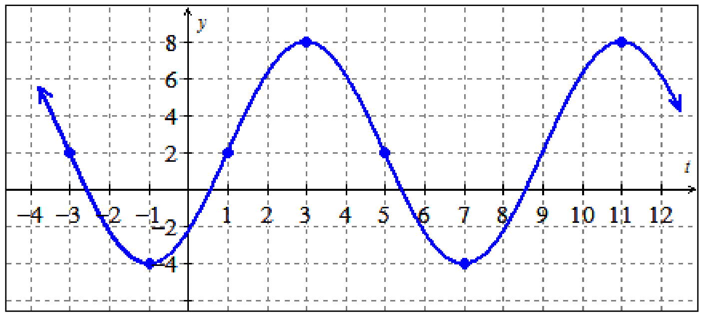
- the amplitude is 3 units
- the midline is $y = 2$
- the period is $2\pi \cdot \frac{1}{\pi} = 2$ units

The horizontal shift is $\frac{1}{2}$ of a unit to the left, i.e., $h = -\frac{1}{2}$, so we'll "start" a cosine wave at $t = -\frac{1}{2}$ and make sure it has the appropriate midline, amplitude, and period.



A graph of $g(t) = 3\cos\left(\pi t + \frac{\pi}{2}\right) + 2$.

7. Find a possible algebraic rule for the sinusoidal function f graphed below. (You only need to provide one algebraic rule and you may utilize either the sine or cosine function.)



The graph of $y = f(t)$.

There are many possibilities. Let's write a rule involving cosine, so our rule will have the form $f(t) = A\cos\big(\omega(t-h)\big) + k$.

Since the amplitude is 6, we know that $A = 6$.

Since the midline is $y = 2$, we know that $k = 2$.

Since the period is 8, we know that $8 = \frac{2\pi}{\omega}$; thus, $\omega = \frac{\pi}{4}$.

Since a natural cosine graph "starts" at $t = 3$, we can use $h = 3$.

Therefore, an algebraic rule for the graphed function is $f(t) = 6\cos\left(\frac{\pi}{4}(t-3)\right) + 2$.

8. Evaluate the following expressions. [Be sure to compose conclusions that *directly* communicate what the given expressions equal and to provide completely simplified exact numerical values.]

a. $\cos\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

$$\cos\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = \cos\left(-\frac{\pi}{3}\right)$$
$$= \frac{1}{2}$$

b. $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right)$

$$\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right) = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$$
$$= -\frac{\pi}{4}$$

9. Find ***all*** of the solutions of the following trigonometric equation. [Provide completely simplified, exact numerical values.]

$$6\sin(2\theta) + 3 = 0$$

$$6\sin(2\theta) + 3 = 0$$

$$\Rightarrow \quad 6\sin(2\theta) = -3$$

$$\Rightarrow \quad \sin(2\theta) = \tfrac{-3}{6} = -\tfrac{1}{2}$$

$$\Rightarrow \quad 2\theta = \tfrac{7\pi}{6} + 2k\pi \quad \text{or} \quad 2\theta = \tfrac{11\pi}{6} + 2k\pi, \; k \in \mathbb{Z}$$

$$\Rightarrow \quad \frac{2\theta}{2} = \frac{\frac{7\pi}{6}}{2} + \frac{2k\pi}{2} \quad \text{or} \quad \frac{2\theta}{2} = \frac{\frac{11\pi}{6}}{2} + \frac{2k\pi}{2}, \; k \in \mathbb{Z}$$

$$\Rightarrow \quad \theta = \tfrac{7\pi}{12} + k\pi \quad \text{or} \quad \theta = \tfrac{11\pi}{12} + k\pi, \; k \in \mathbb{Z}$$

10. Find all of the solutions of the following trigonometric equation **on the interval $[0, 2\pi)$**. [Provide completely simplified, exact numerical values.]

$$2\cos(3x) + 6 = 4$$

$$2\cos(3x) + 6 = 4$$

$$\Rightarrow \quad 2\cos(3x) = -2$$

$$\Rightarrow \quad \cos(3x) = -1$$

$$\Rightarrow \quad 3x = \pi + 2k\pi, \quad k \in \mathbb{Z}$$

Note that we only need one "family" of solutions since the output "-1" only occurs once in each period of $y = \cos(x)$.

$$\Rightarrow \quad \frac{3x}{3} = \frac{\pi}{3} + \frac{2k\pi}{3}$$

$$\Rightarrow \quad x = \tfrac{\pi}{3} + \tfrac{2k\pi}{3}, \quad k \in \mathbb{Z}$$

Since we need $x \in [0, 2\pi)$, we can see use $k = 0$, $k = 1$, and $k = 2$ to obtain the solution set:

$$\left\{ \tfrac{\pi}{3}, \tfrac{\pi}{3} + \tfrac{2\pi}{3}, \tfrac{\pi}{3} + \tfrac{4\pi}{3} \right\} = \left\{ \tfrac{\pi}{3}, \pi, \tfrac{5\pi}{3} \right\}.$$