Section V: Parametric and Implicit Equations

Chapter 3: More on Parametric and Implicit Equations

In this chapter, we will study the parametric and implicit forms of **ellipses** and **hyperbolas**.

ELLIPSES

As we observed in the previous chapter, if we start with the system of parametric equations

$$\begin{cases} x = \cos(t) \\ y = \sin(t). \end{cases}$$

that defines a circle of radius 1 centered at the origin and multiply *both* the x- and y-coordinate by a factor of r and add constants h and k to the x- and y-coordinates, respectively, we obtain the system

$$\begin{cases} x = h + r\cos(t) \\ y = k + r\sin(t) \end{cases}$$

that defines a circle of radius r centered at the point (h, k). What would happen if we didn't multiply the x- and y-coordinate by the same factor r, but instead multiply the x-coordinate by the factor a and the y-coordinate by the factor b where $a \neq b$? If $a > b$, then the x-coordinates will be stretched more than the y-coordinates so we should expect a warped circle, or an oval, that is longer the horizontally than vertically. Similarly, if $b > a$, we should expect an oval that is longer vertically than horizontally. The mathematical term for an oval is **ellipse**.

If $h, k, a, b \in \mathbb{R}$ then the system of parametric equations below defines an ellipse centered at the point (h, k):

$$\begin{cases} x = h + a\cos(t) \\ y = k + b\sin(t). \end{cases}$$

We usually take $a, b > 0$. The **horizontal axis** of the ellipse is $2a$ units and the **vertical axis** is $2b$ units.

EXAMPLE 1: Sketch a graph of the ellipse defined by the system of parametric equations

$$\begin{cases} x = -2 + 2\cos(t) \\ y = 3 + 6\sin(t). \end{cases}$$

SOLUTION:

Based on what we observed above, we see that the center is $(-2, 3)$ and the horizontal axis is $2 \cdot 2 = 4$ units and the vertical axis is $2 \cdot 6 = 12$ units; see Figure 1.

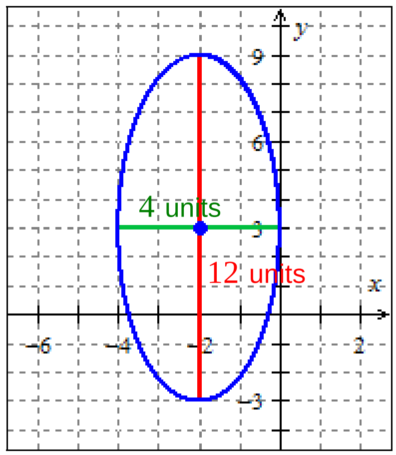

Figure 1: The ellipse defined by

$$\begin{cases} x = -2 + 2\cos(t) \\ y = 3 + 6\sin(t). \end{cases}$$

You should sketch this system on your graphing calculator and make sure you obtain the same graph.

EXAMPLE 2: Eliminate the parameter t from the system of parametric equations

$$\begin{cases} x = -2 + 2\cos(t) \\ y = 3 + 6\sin(t). \end{cases}$$

to obtain an implicit equation that describes this ellipse.

SOLUTION:

We can use the Pythagorean Identity to eliminate the parameter. The Pythagorean Identity involves $\sin(t)$ and $\cos(t)$ so we need to first we need to isolate $\sin(t)$ and $\cos(t)$ in the equations in our system:

$$x = -2 + 2\cos(t) \qquad\qquad \text{and} \qquad\qquad y = 3 + 6\sin(t)$$

$$\Rightarrow 2\cos(t) = x + 2 \qquad\qquad \Rightarrow 6\sin(t) = y - 3$$

$$\Rightarrow \cos(t) = \frac{x+2}{2} \qquad\qquad \Rightarrow \sin(t) = \frac{y-3}{6}$$

Now, we can substitute the expressions $\frac{x+2}{2}$ and $\frac{y-3}{6}$ for $\cos(t)$ and $\sin(t)$ in the Pythagorean Identity and obtain an implicit equation for the ellipse:

$$\begin{aligned} \cos^2(t) + \sin^2(t) &= 1 \\ \Rightarrow \left(\frac{x+2}{2}\right)^2 + \left(\frac{y-3}{6}\right)^2 &= 1 \\ \Rightarrow \frac{(x+2)^2}{2^2} + \frac{(y-3)^2}{6^2} &= 1 \\ \Rightarrow \frac{(x+2)^2}{4} + \frac{(y-3)^2}{36} &= 1 \end{aligned}$$

Thus, the implicit equation $\frac{(x+2)^2}{4} + \frac{(y-3)^2}{36} = 1$ represents the same ellipse defined by the given system of parametric equations.

EXAMPLE 3: The system of parametric equations

$$\begin{cases} x = h + a\cos(t) \\ y = k + b\sin(t) \end{cases}$$

defines an ellipse centered at the point (h, k) with horizontal axis $2a$ units and the vertical axis $2b$ units. Find an implicit equation that describes the same ellipse.

SOLUTION:

We can use the Pythagorean Identity to eliminate the parameter just as we did in Example 2. First we need to isolate $\sin(t)$ and $\cos(t)$ in the equations in our system:

$$x = h + a\cos(t) \qquad\qquad \text{and} \qquad\qquad y = k + b\sin(t)$$

$$\Rightarrow a\cos(t) = x - h \qquad\qquad \Rightarrow b\sin(t) = y - k$$

$$\Rightarrow \cos(t) = \frac{x-h}{a} \qquad\qquad \Rightarrow \sin(t) = \frac{y-k}{b}$$

Now, we can substitute the expressions $\frac{x-h}{a}$ and $\frac{y-k}{b}$ for $\cos(t)$ and $\sin(t)$ in the Pythagorean Identity and obtain an implicit equation for the ellipse:

$$\begin{aligned} \cos^2(t) + \sin^2(t) &= 1 \\ \Rightarrow \left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 &= 1 \\ \Rightarrow \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} &= 1 \end{aligned}$$

If $h, k, a, b \in \mathbb{R}$, then the implicit equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

defines an **ellipse** centered at the point (h, k) with horizontal axis $2a$ units and the vertical axis $2b$ units.

HYPERBOLAS

EXAMPLE 4: Sketch the graph of the system of parametric equations

$$\begin{cases} x = 2\tan(t) \\ y = 3\sec(t) \end{cases}$$

and eliminate the parameter t to obtain an implicit equation that describes the same curve.

SOLUTION:

We can use our graphing calculator (or any other graphing utility) to graph the system; see Figure 2 below.

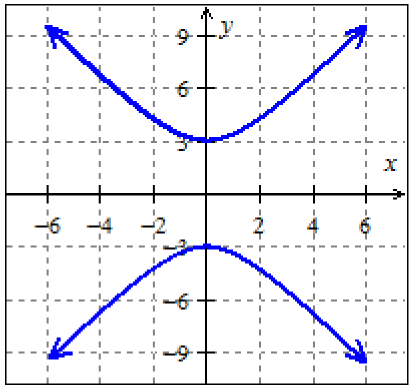

Figure 2: The graph of the parametric system

$$\begin{cases} x = 2\tan(t) \\ y = 3\sec(t) \end{cases}$$

Graphs like the one in Figure 2 are called **hyperbolas**. Hyperbolas have diagonal *asymptotes* that can be found using the "2" and "3" in the rules for the x- and y-coordinates, respectively, to draw a rectangle with horizontal length $2 \cdot 2 = 4$ and vertical length $2 \cdot 3 = 6$, and the asymptotes for the hyperbola are the diagonals of this rectangle. See Figure 3, below.

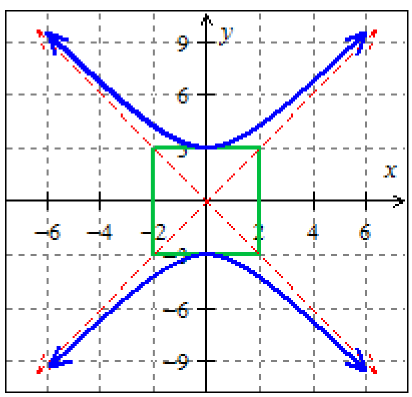

Figure 3

Since the parameterization

$$\begin{cases} x = 2\tan(t) \\ y = 3\sec(t) \end{cases}$$

involves tangent and secant, we can use the identity $\sec^2(t) - \tan^2(t) = 1$ to eliminate the parameter t and obtain an implicit equation. (We first saw this version of the Pythagorean

Identity in Section I: Chapter 3.) In order to utilize this identity, we need to solve the equations involved in our parameterization for $\tan(t)$ and $\sec(t)$, respectively:

$$x = 2\tan(t) \qquad \text{and} \qquad y = 3\sec(t)$$

$$\Rightarrow \tan(t) = \frac{x}{2} \qquad \qquad \Rightarrow \sec(t) = \frac{y}{3}$$

Now we can substitute $\frac{x}{2}$ and $\frac{y}{3}$ for $\tan(t)$ and $\sec(t)$ in the identity $\sec^2(t) - \tan^2(t) = 1$:

$$\begin{aligned} &\sec^2(t) - \tan^2(t) = 1 \\ \Rightarrow\ &\left(\frac{y}{3}\right)^2 - \left(\frac{x}{2}\right)^2 = 1 \\ \Rightarrow\ &\frac{y^2}{9} - \frac{x^2}{4} = 1 \end{aligned}$$

Thus, the implicit equation $\frac{y^2}{9} - \frac{x^2}{4} = 1$ describes the same hyperbola as the system of parametric equations

$$\begin{cases} x = 2\tan(t) \\ y = 3\sec(t). \end{cases}$$

EXAMPLE 5: Sketch the graph of the system of parametric equations

$$\begin{cases} x = 2 + 4\sec(t) \\ y = -3 + 3\tan(t) \end{cases}$$

and eliminate the parameter t to obtain an implicit equation that represents the same curve.

SOLUTION:

We should expect the center of the hyperbola to be shifted to the right 2 units and down 3 units "2" and "–3" in the rules for the x- and y-coordinates, respectively. Also, we should expect the asymptotes for the hyperbola to the diagonals of a rectangle with horizontal length $2 \cdot 4 = 8$ and vertical length $2 \cdot 3 = 6$. Below is the graph of this system.

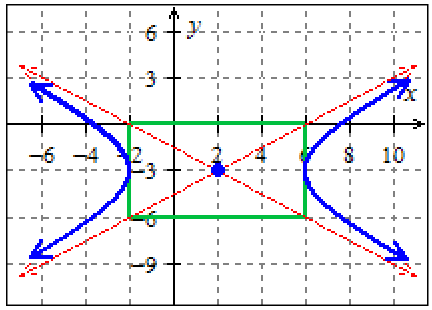

Figure 4

Obviously, this hyperbola opens towards the horizontal direction while the hyperbola we studied in Example 4 opened in the vertical direction. The reason that these hyperbolas open in different directions is that in Example 4 tangent is involved in the rule for the x-coordinate and secant is involved in the rule for the y-coordinate but in this example secant is involved in the rule for the x-coordinate and tangent is involved in the rule for the y-coordinate.

Let's find an implicit equation that describes the hyperbola in Figure 4 and observe how the implicit equations for the hyperbolas that open horizontally differ from those that open vertically. As we did in Example 4, we can use the identity $\sec^2(t) - \tan^2(t) = 1$ to eliminate the parameter t and obtain an implicit equation. First, let's solve the equations involved in our parameterization for $\tan(t)$ and $\sec(t)$, respectively:

$$x = 2 + 4\sec(t) \qquad\qquad y = -3 + 3\tan(t)$$

$$\Rightarrow \sec(t) = \frac{x-2}{4} \qquad \text{and} \qquad \Rightarrow \tan(t) = \frac{y+3}{3}$$

Now we can substitute $\frac{x-2}{4}$ and $\frac{y+3}{3}$ for $\sec(t)$ and $\tan(t)$ in the identity $\sec^2(t) - \tan^2(t) = 1$:

$$\sec^2(t) - \tan^2(t) = 1$$

$$\Rightarrow \left(\frac{x-2}{4}\right)^2 - \left(\frac{y+3}{3}\right)^2 = 1$$

$$\Rightarrow \frac{(x-2)^2}{16} - \frac{(y+3)^2}{9} = 1$$

Thus, the implicit equation $\frac{(x-2)^2}{16} - \frac{(y+3)^2}{9} = 1$ describes the same hyperbola as the given system of parametric equations. Notice that unlike the implicit equation we found in Example 4 in which the has the from "*an expression involving y minus an expression involving x*", this equation has the form "*an expression involving x minus an expression involving y*"; it is this difference that makes the two hyperbolas to open in different directions.

If $h, k, a, b \in \mathbb{R}$ then the system of parametric equations

$$\begin{cases} x = h + a\tan(t) \\ y = k + b\sec(t) \end{cases}$$

defines a **hyperbola** that is centered at the point (h, k) and opens in the *vertical direction*; the system

$$\begin{cases} x = h + a\sec(t) \\ y = k + b\tan(t) \end{cases}$$

defines a **hyperbola** that is centered at the point (h, k) and opens in the *horizontal* direction. The asymptote of the hyperbola can be found by drawing the diagonals of a rectangle centered at the point (h, k) with horizontal length $2a$ units and vertical length $2b$ units.

If $h, k, a, b \in \mathbb{R}$, then the implicit equation

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

defines a **hyperbola** that is centered at the point (h, k) and opens in the *vertical* direction while implicit equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

defines a **hyperbola** that is centered at the point (h, k) and opens in the *horizontal* direction. The asymptotes of the hyperbola can be found by drawing the diagonals of a rectangle centered at the point (h, k) with horizontal length $2a$ units and vertical length $2b$ units.