Section V: Parametric and Implicit Equations

Chapter 2: Introduction to Implicit Equations

When we describe curves on the coordinate plane with algebraic equations, we can define the relationship between the x- and y-coordinates of the curve either *explicitly* or *implicitly*. To explain the difference, let's consider the equation $y = 3x + 7$.

- The equation $y = 3x + 7$ establishes a relationship between x and y. We say that y is defined ***explicitly*** in terms of x since the equation gives us a very clear description of the relationship between x and y: no matter what the x-value is, the corresponding y-value is always $3x + 7$.

- If we subtract $3x$ from each side of this equation we obtain the equation $-3x + y = 7$. Clearly this equation describes the same relationship between x and y as the equation $y = 3x + 7$, but when y isn't isolated, the relationship between the variables isn't explicit. Instead, this relationship is *implied* by the equation, so we say that it is an ***implicit equation***.

An equation involving two variables is considered *explicit* if one of the variables is isolated on one side of the equation, while and equation is considered *implicit* if neither of the variables is isolated on one side of the equation. As we've seen, equations like $y = 3x + 7$ can be written either explicitly or implicitly, but many equations can only be written implicitly. For example, it's not possible to solve the implicit equation $x^2 + y^2 = 1$ for either variable. If we try to solve the equation for y, we need to use two explicit equations to communicate the information in the implicit equation:

$$
\begin{aligned}
& x^2 + y^2 = 1 \\
\Rightarrow\quad & y^2 = 1 - x^2 \\
\Rightarrow\quad & y = \pm\sqrt{1 - x^2} \\
\Rightarrow\quad & y = \sqrt{1 - x^2} \quad \text{or} \quad y = -\sqrt{1 - x^2}
\end{aligned}
$$

EXAMPLE 1: Recall from Example 4 in the previous chapter that the system of parametric equations below describe a unit circle centered at the point $(0,\, 0)$:

$$\begin{cases} x = \cos(t) \\ y = \sin(t). \end{cases}$$

As we learned in the previous chapter, we use the Pythagorean Theorem **to eliminate the parameter t** and transform this system into a single implicit equation:

$$\begin{aligned} &\cos^2(t) + \sin^2(t) = 1 \\ \Rightarrow\ &\ x^2 \quad + \quad y^2 \quad = 1 \end{aligned}$$

Thus, the implicit equation $x^2 + y^2 = 1$ describes a unit circle centered at the point $(0,\, 0)$.

Since the parametric system

$$\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases}$$

describes a circle of radius 1 unit, we should expect that if we multiply *both* the x- and y-coordinate by the factor r we will stretch (or compress) the circle so that its radius will change from 1 unit to r units, and if we add constants h and k to the x- and y-coordinates, respectively, we will shift the center of the circle from the point $(0,\, 0)$ to the point $(h,\, k)$. A generalized parameterization of a circle is given in the box below.

> If $h, k, r \in \mathbb{R}$ and $r > 0$ then the system of parametric equations below defines a **circle** of radius r centered at the point $(h,\, k)$.
>
> $$\begin{cases} x = h + r\cos(t) \\ y = k + r\sin(t) \end{cases}$$

EXAMPLE 2: Eliminate the parameter t from the system of parametric equations given in the box above to obtain an implicit equation that defines a circle of radius r centered at the point (h, k).

SOLUTION:

We can use the Pythagorean Theorem to eliminate the parameter just as we did in Example 1. The Pythagorean Theorem involves $\sin(t)$ and $\cos(t)$ so we need to first we need to isolate $\sin(t)$ and $\cos(t)$ in the equations in our system:

$$\begin{aligned} & x = h + r\cos(t) \\ \Rightarrow\ & r\cos(t) = x - h \\ \Rightarrow\ & \cos(t) = \frac{x-h}{r} \end{aligned} \qquad \text{and} \qquad \begin{aligned} & y = k + r\sin(t) \\ \Rightarrow\ & r\sin(t) = y - k \\ \Rightarrow\ & \sin(t) = \frac{y-k}{r} \end{aligned}$$

Now, we can substitute the expressions $\frac{x-h}{r}$ and $\frac{y-k}{r}$ for $\cos(t)$ and $\sin(t)$ in the Pythagorean Theorem and obtain an implicit equation for the circle:

$$\begin{aligned} & \cos^2(t) + \sin^2(t) = 1 \\ \Rightarrow\ & \left(\frac{x-h}{r}\right)^2 + \left(\frac{y-k}{r}\right)^2 = 1 \\ \Rightarrow\ & \frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{r^2} = 1 \\ \Rightarrow\ & (x-h)^2 + (y-k)^2 = r^2 \end{aligned}$$

If $h, k, r \in \mathbb{R}$ and $r > 0$ then the implicit equation

$$(x-h)^2 + (y-k)^2 = r^2$$

defines a **circle** of radius r centered at the point (h, k).

EXAMPLE 3: Determine the center and radius of the circle defined by the implicit equation $(x+7)^2+(y-6)^2=9$.

SOLUTION:

Notice that the equation $(x+7)^2+(y-6)^2=9$ has the form $(x-h)^2+(y-k)^2=r^2$ where $h=-7$, $k=6$, and $r=3$. Thus, the center of the circle is at the point $(-7, 6)$ and the radius is 3 units. We could check if we are correct by graphing the circle on our graphing calculator, but our calculators aren't able to graph implicit equations. If we transform our implicit equation into a system of parametric equations, we could use our graphing calculator to graph the circle.

To transform the implicit equation into a system of parametric equations, we mimic what we did in Example 2 but do everything in the opposite order.

First, let's get 1 on the right side of the equation by dividing both sides by 3^2 (i.e., 9):

$$\begin{aligned} & (x+7)^2+(y-6)^2=9 \\ \Rightarrow\quad & (x+7)^2+(y-6)^2=3^2 \\ \Rightarrow\quad & \frac{(x+7)^2}{3^2}+\frac{(y-6)^2}{3^2}=1 \\ \Rightarrow\quad & \left(\frac{x+7}{3}\right)^2+\left(\frac{y-6}{3}\right)^2=1 \qquad \langle a\rangle \end{aligned}$$

We now have an equation in which the sum of two squares is equal to 1. Since Pythagorean Theorem has this same form (i.e., since $\cos^2(t)+\sin^2(t)=1$), we can let $\cos(t)=\dfrac{x+7}{3}$ and $\sin(t)=\dfrac{y-6}{3}$. (Notice that if we substitute these values for $\cos(t)$ and $\sin(t)$ in the Pythagorean Theorem we obtain the equation labeled $\langle a\rangle$ above.) We can solve these equations for x and y to obtain a parameterization of the given implicit equation:

$$\begin{aligned} & \cos(t)=\frac{x+7}{3} \\ \Rightarrow\quad & x+7=3\cos(t) \\ \Rightarrow\quad & x=-7+3\cos(t) \end{aligned} \qquad \text{and} \qquad \begin{aligned} & \sin(t)=\frac{y-6}{3} \\ \Rightarrow\quad & y-6=3\sin(t) \\ \Rightarrow\quad & y=6+3\sin(t) \end{aligned}$$

Thus, the system of parametric equations below describes the same circle as the implicit equation $(x+7)^2+(y-6)^2=9$.

$$\begin{cases} x=-7+3\cos(t) \\ y=6+3\sin(t) \end{cases}$$

Notice that from the generalization given at the beginning of Example 2, we can see that this parameterization describes a circle of radius 3 units with center $(-7, 6)$ which is exactly what we were looking for.

EXAMPLE 4: The graph of the equation $x^2 + y^2 - 6x + 4y = 3$ is a circle. Identify the center and radius (by using appropriate algebra).

SOLUTION:

In order to determine the center and radius of the circle, we need to get our implicit equation into the form

$$(x - h)^2 + (y - k)^2 = r^2 .$$

To accomplish this, we can complete the square *twice*. (If you need to review the procedure for *completing the square*, see page A40 in the appendix of our textbook as well as page 144 in §2.3 in our textbook.)

$$\begin{aligned}
& x^2 + y^2 - 6x + 4y = 3 \\
\Rightarrow\quad & x^2 - 6x + y^2 + 4y = 3 \\
\Rightarrow\quad & x^2 - 6x + 9 - 9 + y^2 + 4y + 4 - 4 = 3 \\
\Rightarrow\quad & (x^2 - 6x + 9) + (y^2 + 4y + 4) - 9 - 4 = 3 \\
\Rightarrow\quad & (x - 3)^2 + (x + 2)^2 - 13 = 3 \\
\Rightarrow\quad & (x - 3)^2 + (x + 2)^2 = 16 \\
\Rightarrow\quad & (x - 3)^2 + (x + 2)^2 = 4^2
\end{aligned}$$

Thus, the center of the circle is at the point $(3, -2)$ and the radius of the circle is 4 units. Based on what we observed in Examples 3 and 4, we can conclude that the system of parametric equations below describes this same circle.

$$\begin{cases} x = 3 + 4\cos(t) \\ y = -2 + 4\sin(t) \end{cases}$$

Graph the system on your graphing calculator to check.