Section V: Parametric and Implicit Equations

Chapter 1: Parametric Equations

EXAMPLE 1: Consider a figure skater carving a figure-eight on the ice. Figure 1 below shows the skater's path; we've embedded a coordinate plane on the ice-rink.

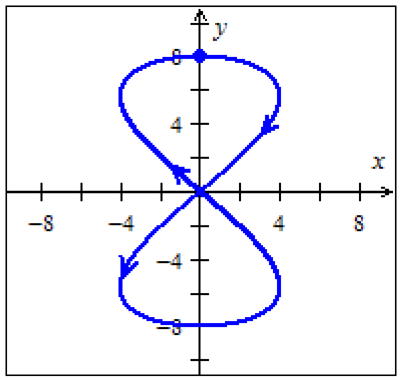


Figure 1: The skater's figure-eight.

Clearly, the y-values are not a function of the x-values (since this graph fails the "vertical line test"). But it should also be clear that at each moment in time, the skater is in *exactly* one location. In other words, the skater's location ***is*** a function of time. We can use a system of **parametric equations** to define the skater's location as a function of time. A system of parametric equations consists of a pair of functions: one that describes the x-coordinate and the other that describes the y-coordinate; typically, the input variable for these functions represents *time*. It turns out that the system of parametric equations given below describes the skater's motion that we've graphed in Figure 1 above:

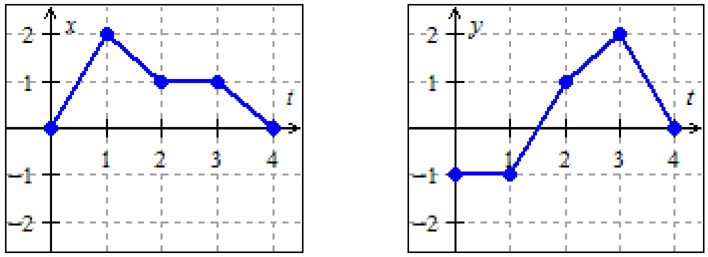
$$\begin{cases} x(t) = 4\sin\left(\frac{\pi}{2}t\right) \\ y(t) = 8\cos\left(\frac{\pi}{4}t\right) \end{cases}$$

Let's sketch a graph of this system of parametric equations for $0 \le t \le 8$. (Let's assume that t is measured in seconds, so we're graphing the figure skater's movement during her first 8 seconds of skating.) Note that we should expect to obtain the same graph as given in Figure 1, but we need to study how to obtain such a graph ourselves.

CLICK HERE to see the system of parametric equations that describe the figure skater's path graphed by hand.

Note that if you want to graph this function on your calculator, you need to change the *graph mode* of your calculator to "**parametric**".

EXAMPLE 2: Suppose that a robot is moving around a coordinate plane and that $x = f(t)$ represents that x-coordinate and $y = g(t)$ represents the y-coordinate of the robots location on the plane as functions of time, t, in seconds. The graphs of $x = f(t)$ and $y = g(t)$ during the first four minutes of the robot's travels are given in Figure 2; sketch a graph of the robot's movement.



The graph of $x = f(t)$. The graph of $y = g(t)$.

Figure 2

SOLUTION:

To graph the movement of the robot, we need to find and plot ordered pairs $(x, y) = \left(f(t), g(t) \right)$. To find these ordered pairs, we need to choose values for t and find the corresponding values of $x = f(t)$ and $y = g(t)$.

$t = 0$: $\left. \begin{aligned} x &= f(0) = 0 \\ y &= g(0) = -1 \end{aligned} \right\}$ So the robot is at $(0, -1)$ when $t = 0$.

$t = 1$: $\left. \begin{aligned} x &= f(1) = 2 \\ y &= g(1) = -1 \end{aligned} \right\}$ So the robot is at $(2, -1)$ when $t = 1$.

$t = 2$: $\left. \begin{aligned} x &= f(2) = 1 \\ y &= g(2) = 1 \end{aligned} \right\}$ So the robot is at $(1, 1)$ when $t = 2$.

$t = 3$: $\left. \begin{aligned} x &= f(3) = 1 \\ y &= g(3) = 2 \end{aligned} \right\}$ So the robot is at $(1, 2)$ when $t = 3$.

$t = 4$: $\left. \begin{aligned} x &= f(4) = 0 \\ y &= g(4) = 0 \end{aligned} \right\}$ So the robot is at $(0, 0)$ when $t = 4$.

Now we can plot these five points and connect them in order, starting with the point representing $t = 0$ and ending with the point representing $t = 4$. (See Figure 3 below.) We use arrows to keep track of the direction of travel.

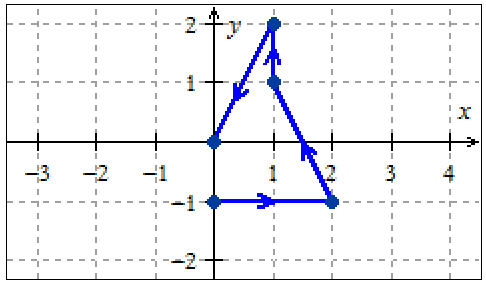


Figure 3: The graph of $(x,\ y) = \left(f(t),\ g(t) \right)$.

EXAMPLE 3: Suppose that the x- and y-coordinates of the movement of a robot are given by the following functions of time, t, in seconds. (We'll assume that these equations apply for all $t \geq 0$.)

$$\begin{cases} x = 6 - 2t \\ y = -2 + 4t. \end{cases}$$

Sketch a graph of the robot's movement.

SOLUTION:

To graph the movement of the robot, we need to find and plot ordered pairs $(x,\ y)$. To find these ordered pairs, we need to choose values for t and find the corresponding values of x and y using the parametric equations given above.

$$t = 0: \quad \left. \begin{aligned} x &= 6 - 2(0) \\ &= 6 \\ y &= -2 + 4(0) \\ &= -2 \end{aligned} \right\} \quad \text{So the robot is at } (6, -2) \text{ when } t = 0.$$

$$t = 1: \quad \left. \begin{aligned} x &= 6 - 2(1) \\ &= 4 \\ y &= -2 + 4(1) \\ &= 2 \end{aligned} \right\} \quad \text{So the robot is at } (4,\ 2) \text{ when } t = 1.$$

$$t = 2: \quad \left. \begin{aligned} x &= 6 - 2(2) \\ &= 2 \\ y &= -2 + 4(2) \\ &= 6 \end{aligned} \right\} \quad \text{So the robot is at } (2,\ 6) \text{ when } t = 2.$$

$$\left.\begin{aligned} t = 3: \quad x &= 6 - 2(3) \\ &= 0 \\ y &= -2 + 4(3) \\ &= 10 \end{aligned}\right\} \text{ So the robot is at } (0, 10) \text{ when } t = 3.$$

Now we can plot these points and connect them in order, starting with the point representing $t = 0$. (See Figure 4.) We use arrows to keep track of the direction of travel, and to suggest the direction that the robot will travel in the future.

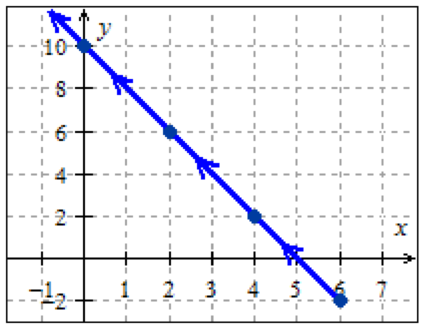


Figure 4: The robot's movement for $t \geq 0$.

Clearly, the robot is moving linearly. We can **eliminate the parameter *t*** to find a single equation that describes the robot's movement, and we should expect our equation to be linear.

To eliminate the parameter t, we can solve one of the equations in our system

$$\begin{cases} x = 6 - 2t \\ y = -2 + 4t. \end{cases}$$

for t and then substitute the result into our other equation. Let's solve the equation that represents the x-coordinate for t and substitute into the equation that represents the y-coordinate:

$$\begin{aligned} x &= 6 - 2t \\ \Rightarrow \quad 2t &= 6 - x \\ \Rightarrow \quad t &= \tfrac{1}{2}(6 - x). \end{aligned}$$

So

$$\begin{aligned} y &= -2 + 4t \\ &= -2 + 4\left(\tfrac{1}{2}(6 - x)\right) \\ &= -2 + 2(6 - x) \\ &= 10 - 2x. \end{aligned}$$

Thus, path that the robot follows is represented by the equation $y = 10 - 2x$. (Notice that this equation does in fact represent the line we graphed above.) Although this *single* equation is arguably more efficient than the *system* of parametric equations, the advantage of the system of parametric equations is that, in addition to describing the path that the robot travels, it tells us WHEN the robot is in at each point along the path.

A system of parametric equations involves **three** variables so it conveys three pieces of information (rather than just the two pieces of information conveyed by a single equation in two-variables). Typically, the parameter t represents time so that in addition to describing the path that an object travels, a system of parametric equations also describes the time that it takes the object to travel along the path.

EXAMPLE 4: The x- and y-coordinates of the movement of a particle are given by the following functions of time, t, where $0 \le t \le 2\pi$:

$$\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases}.$$

The graph of this system is given in Figure 5. (You should graph it on your graphing calculator and make sure that you are able to obtain the same graph.)

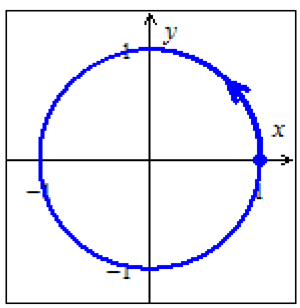


Figure 5: The graph of $\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases}$.

Let's eliminate the parameter t from this system in order to find a single equation involving x and y that represents the path (i.e., the unit circle) that the particle travels along. In this case, rather than using a substitution technique like we used in the previous example, we'll utilize a trigonometric identity. (We'll use an identity since identities are ALWAYS true, no matter what the values of the variables.) In this case, we need an identity that involves $\cos(t)$ and $\sin(t)$ since these expressions are involved in our system of parametric equation. Let's use the Pythagorean Identity: $\sin^2(t) + \cos^2(t) = 1$, and substitute x and y for $\cos(t)$ and $\sin(t)$, respectively:

$$\begin{aligned} &\sin^2(t) + \cos^2(t) = 1 \\ \Rightarrow\quad &y^2 + x^2 = 1 \end{aligned}$$

Thus, the equation $x^2 + y^2 = 1$ describes the path that the particle travels along, which is what we should have expected since the path is a unit circle! Note that the drawback to this equation is that, unlike the system of parametric equations, it doesn't tell us WHEN the particle is at each point.

EXAMPLE 5: The x- and y-coordinates of the movement of a particle are given by the following functions of time, t, where $0 \le t \le 2\pi$:

$$\begin{cases} x = \cos(t) - 1 \\ y = 2\sin(t) + 2. \end{cases}$$

The graph of this system is given in Figure 6. (You should graph it on your graphing calculator and make sure that you are able to obtain the same graph.)

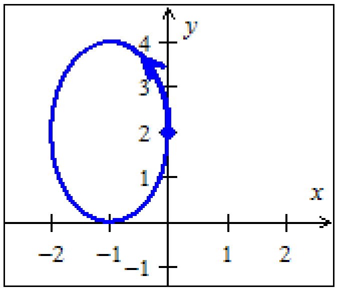


Figure 6: The graph of the particle's movement.

Let's eliminate the parameter t from this system in order to find a single equation involving x and y that represents the path that the particle travels along. As we did in the previous example, we can use the Pythagorean Identity: $\sin^2(t) + \cos^2(t) = 1$. But first we need to solve the equations in our system for $\cos(t)$ and $\sin(t)$ so that we can then substitute these expressions for $\cos(t)$ and $\sin(t)$ in the Pythagorean Identity:

$$x = \cos(t) - 1 \quad \Rightarrow \quad \cos(t) = x + 1$$

and

$$y = 2\sin(t) + 2 \quad \Rightarrow \quad \sin(t) = \tfrac{1}{2}(y - 2).$$

Thus,

$$\begin{aligned} \sin^2(t) + \cos^2(t) &= 1 \\ \Rightarrow \quad (x+1)^2 + \left(\tfrac{1}{2}(y-2)\right)^2 &= 1 \\ \Rightarrow \quad (x+1)^2 + \tfrac{1}{4}(y-2)^2 &= 1. \end{aligned}$$

So the equation $(x+1)^2 + \frac{1}{4}(y-2)^2 = 1$ describes the path that the particle travels along. In the next two chapters we'll study *implicit* equations like this one. This equation describes an **ellipse** (i.e., an oval).

EXAMPLE 6: Recall that the equation $r = \theta$ in polar coordinates represents the Archimedean spiral. Express this spiral in parametric equations.

SOLUTION:

To parameterize the polar equation $r = \theta$ we need too find two functions: one that represents the x-coordinate and one that represents the y-coordinate. In Section III, Chapter 1, we found that the polar coordinates (r, θ) can be translated into rectangular coordinates (x, y) by using the following formulas:

$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta)$$

We can use these formulas to parameterize the equation $r = \theta$. The equation tells us that we can substitute θ for r since they are equal:

$$\begin{aligned} x &= r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta) \\ \Rightarrow x &= \theta\cos(\theta) \quad \text{and} \quad y = \theta\sin(\theta). \end{aligned}$$

Finally, by substituting t for θ (since t is the variable we usually use for our parameter), we obtain the following parameterization of the polar equation $r = \theta$:

$$\begin{cases} x = t\cos(t) \\ y = t\sin(t) \end{cases}$$

Be sure to graph this system of parametric equations on your graphing calculator to verify that these equations give us the Archimedean spiral.