# Section I: The Trigonometric Functions

## Chapter 8: Trig Equations and Inverse Trig Functions

**EXAMPLE 1:** Solve the equations below:

**a.** $\sin(t) = \frac{1}{2}$ **b.** $\sin(t) = -0.555$

SOLUTION:

**a.** Based on our experience with the sine function, we know that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, so we know that $t = \frac{\pi}{6}$ is a solution to $\sin(t) = \frac{1}{2}$. We also know that the sine function is periodic with period $2\pi$, so it's values repeat every $2\pi$ units. Thus, $t = \frac{\pi}{6} + 2\pi$ is also a solution to $\sin(t) = \frac{1}{2}$. In fact, $t = \frac{\pi}{6} + 2k\pi$ is a solution for *every* integer $k$. Since $t = \frac{\pi}{6} + 2k\pi$ is a solution for *every* integer, this represents *infinitely many solutions*, but it still doesn't represent all of the solutions; see Figure 1.

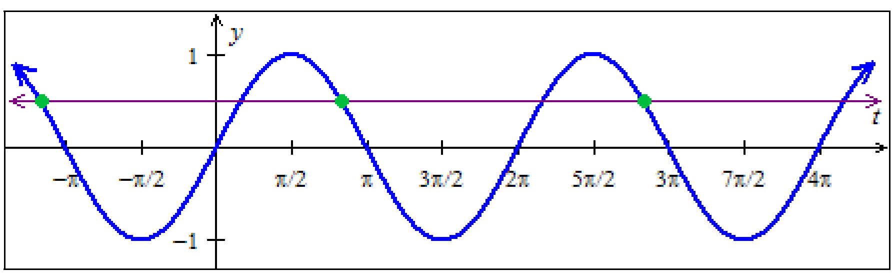


**Figure 1:** The graph of $y = \sin(t)$ and the line $y = \frac{1}{2}$. The green dots represent points with horizontal coordinates of the form $t = \frac{\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$. The other instances where the blue and purple graphs intersect are also solutions to the equation $\sin(t) = \frac{1}{2}$ but they are NOT represented by $t = \frac{\pi}{6} + 2k\pi$.

It should be clear after studying Figure 1 that we are missing lots of solutions. Notice that one of the solutions we are missing is just as close to $\pi$ as our original solution, $t = \frac{\pi}{6}$, is to 0. (Recall the identity $\sin(t) = \sin(\pi - t)$ that we first noticed in Chapter 3.)

Thus, $t = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ is a solution, and the rest of the solutions have the form $t = \frac{5\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$; see Figure 2. So the complete solution is

$$t = \frac{\pi}{6} + 2k\pi \quad \text{for all } k \in \mathbb{Z}$$

or

$$t = \frac{5\pi}{6} + 2k\pi \quad \text{for all } k \in \mathbb{Z}$$

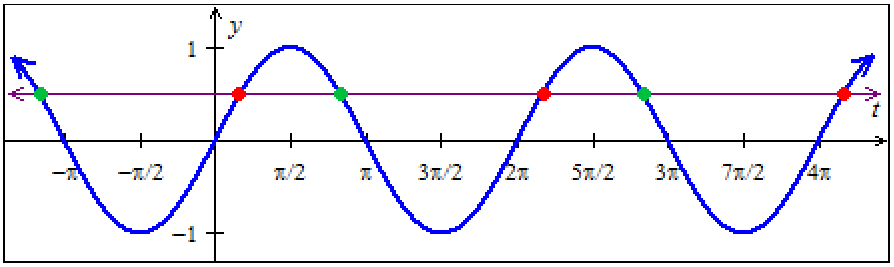


**Figure 2:** The red dots represent points with horizontal coordinates of the form $t = \frac{\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$, while the green dots represent points with horizontal coordinates of the form $t = \frac{5\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$. The orange and green dots together represent all of the solutions to $\sin(t) = \frac{1}{2}$.

**b.** Unlike in part **a**, we don't know what input for $\sin(t)$ is related to the output $-0.555$. Clearly the solutions to $\sin(t) = -0.555$ are not "standard values" for which we have memorized the sine values. In order to solve $\sin(t) = -0.555$, we need a way to "undo" sine. To clarify this concept, let's consider an analogous situation: How do we solve $x^3 = 10$?

In order to solve $x^3 = 10$ we need a way to "undo" cubing. In this case, we can use the cube-root:

$$\begin{aligned} x^3 &= 10 \\ \Rightarrow \sqrt[3]{x^3} &= \sqrt[3]{10} \\ \Rightarrow \quad x &= \sqrt[3]{10} \end{aligned}$$

Notice that the cube-root function is the *inverse* of the cubing function; of course this makes sense since inverse functions "undo" each other. (Inverse functions are studied in MTH 111.) So we need to construct an inverse for the sine function in order to solve equations like $\sin(t) = -0.555$.

The reason we need to "construct" the inverse of the sine function is that the sine function isn't one-to-one since it does not pass the horizontal line test; in Figure 1, above, notice how the graph of $y = \sin(t)$ intersects the horizontal line $y = \frac{1}{2}$ many times. Since $y = \sin(t)$ isn't one-to-one, it doesn't have an inverse function.

Since we *really* want an inverse sine function so that we can solve equations like $\sin(t) = -0.555$, we will restrict the domain of the sine function so that the result is a one-to-one function. We want to choose an interval of the domain that contains a one-to-one portion of the graph, and we want to choose and interval that utilizes the entire range of the sine function. Following tradition, we well choose the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. In Figure 3, this interval of the sine function is highlighted; notice that this on this interval, the function is one-to-one and has the same range as the complete sine function.

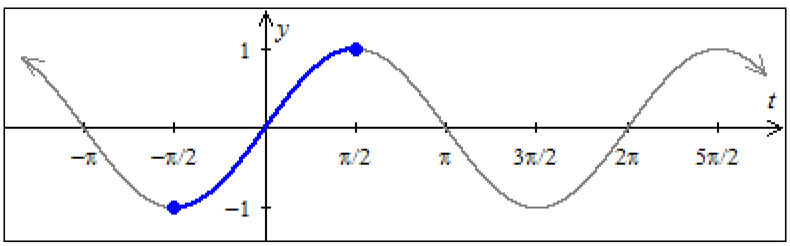


**Figure 3:** The interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ of the graph of $y = \sin(t)$; on this interval, the sine function is one-to-one and has the same range as the complete sine function.

Recall that when we construct the inverse of a function we need to reverse the rolls of the inputs and the outputs, so that the inputs for the origin function become the outputs for the inverse function, and the outputs for the original become the inputs for the inverse.

**DEFINITION:** The **inverse sine function**, denoted $y = \sin^{-1}(t)$, is defined by the following:

If $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and $\sin(y) = t$, then $y = \sin^{-1}(t)$.

By construction, the range of $y = \sin^{-1}(t)$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and the domain is the same as the range of the sine function: $[-1, 1]$. Note that the inverse sine function is often called the **arcsine function** and denoted $y = \arcsin(t)$

**Key Point:** As we've discussed in Chapter 3, we can denote powers of trigonometric functions by putting the exponent between the function name and the input variable; for example, $(\sin(t))^2 = \sin^2(t)$. The definition above implies that inverse function notation looks like the sine function raised to the $-1$ power (i.e., the reciprocal of the sine function), but the reciprocal of a function isn't the same as it's inverse! In order to avoid ambiguous notation, the notation $\sin^{-1}(t)$ *always* refers to the inverse function. If you want to denote the reciprocal of the sine function, you need to use the notation "$(\sin(t))^{-1}$":

$$(\sin(t))^{-1} = \frac{1}{\sin(t)} = \csc(t) \quad \text{but} \quad \csc(t) \neq \sin^{-1}(x)\,!$$

Now we can solve $\sin(t) = -0.555$ and finish part **b** of the Example 1. We were trying to solve this equation when we realized that we needed to construct the inverse sine function. We now possess the tools we'll need to solve the equation, so let's solve it:

$$
\begin{aligned}
& \sin(t) = -0.555 \\
\Rightarrow \quad & \sin^{-1}(\sin(t)) = \sin^{-1}(-0.555) \quad \text{Apply the sine inverse function to both sides} \\
\Rightarrow \quad & t = \sin^{-1}(-0.555) \\
\Rightarrow \quad & t \approx -0.588
\end{aligned}
$$

(Note that we can use a calculator to obtain an approximation for $\sin^{-1}(-0.555)$; you should find a button on your calculator labeled "$\sin^{-1}$".)

Although we've found a solution to the equation, **we aren't don't yet!** Since it's one-to-one, the sine inverse function only gives us one value, but we know that the period nature of the sine function suggests that there are infinitely many solutions to an equation like this. (See Figure 4; notice how many times the sine function reaches the output $-0.555$.)

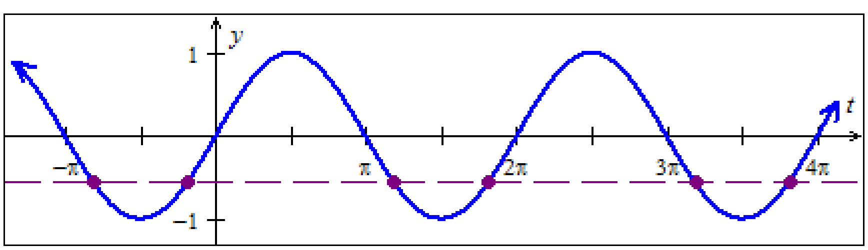


**Figure 4:** The graph of $y = \sin(t)$ intersecting the line $y = -0.555$ many, many times. Each point of intersection represents a solution to $\sin(t) = -0.555$.

We can find all of the solutions by using the solution the inverse sine function gave us as well as the fact that the sine function has period $2\pi$. Since the sine function has period $2\pi$ units, we know that the outputs repeat every $2\pi$ units. So if $t \approx -0.588$ is a solution, $t \approx -0.588 + 2k\pi$ for all $k \in \mathbb{Z}$ must also be solutions. This gives us LOTS of solutions, but we are still missing *half* of them. (Recall we had the same problem in part **a** of the first example in this chapter.) In order to get the rest of the solutions, can use the identity $\sin(t) = \sin(\pi - t)$, and subtract our original solution $(t \approx -0.588)$ from $\pi$: $t \approx \pi - (-0.588) + 2k\pi$, $k \in \mathbb{Z}$. Thus, the complete solution is

$$t \approx -0.588 + 2k\pi \quad \text{for all } k \in \mathbb{Z}$$

or

$$t \approx \pi + 0.588 + 2k\pi \quad \text{for all } k \in \mathbb{Z}$$

**EXAMPLE 2:** Solve $\cos(t) = 0.4$.

SOLUTION:

Now we need a way to "undo" cosine so that we can solve this equation as we solved $\sin(t) = -0.555$ in Example 1. We need an inverse cosine function but, like the sine function, cosine is NOT one-to-one so it doesn't have an inverse function. (See Figure 5.)

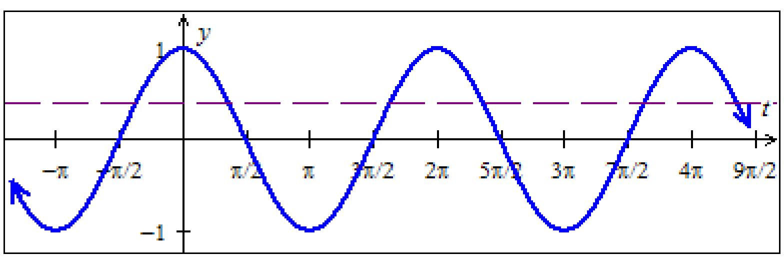


**Figure 5:** The graph of $y = \cos(t)$ and the line $y = 0.4$. Clearly, the cosine function is not one-to-one.

Since we *really* want an inverse cosine function, we will restrict the domain of the cosine function so that the result is a one-to-one function. We want to choose an interval of the domain that contains a one-to-one portion of the graph, and we want to choose and interval that utilizes the entire range of the cosine function. Following tradition, we well choose the interval $[0, \pi]$. In Figure 6 (below), this interval of the cosine function is highlighted; notice that this on this interval, the function is one-to-one and has the same range as the complete cosine function.

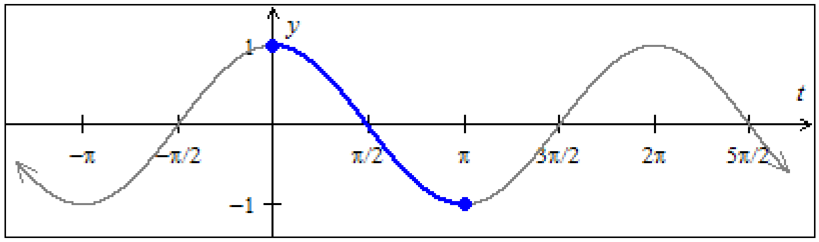


**Figure 6:** The interval $[0, \pi]$ of the graph of $y = \cos(t)$; on this interval, the cosine function is one-to-one and has the same range as the complete cosine function.

Again, recall that when we construct the inverse of a function we need to reverse the rolls of the inputs and the outputs, so that the inputs for the origin function become the outputs for the inverse function, and the outputs for the original become the inputs for the inverse.

**DEFINITION:** The **inverse cosine function**, denoted $y = \cos^{-1}(t)$, is defined by the following:

If $0 \le y \le \pi$ and $\cos(y) = t$, then $y = \cos^{-1}(t)$.

By construction, the range of $y = \cos^{-1}(t)$ is $[0, \pi]$, and the domain is the same as the range of the cosine function: $[-1, 1]$. Note that the inverse cosine function is often called the **arccosine function** and denoted $y = \arccos(t)$

Now we can finish Example 2 by solving $\cos(t) = 0.4$:

$$\begin{aligned} & \cos(t) = 0.4 \\ \Rightarrow\ & \cos^{-1}\big(\cos(t)\big) = \cos^{-1}(0.4) \\ \Rightarrow\ & t = \cos^{-1}(0.4) \\ \Rightarrow\ & t \approx 1.16 \end{aligned}$$

(Note that we can use a calculator to obtain an approximation for $\cos^{-1}(0.4)$; you should find a button on your calculator labeled "$\cos^{-1}$".)

Although we have found a solution to the given equation, **we aren't don't yet!** Since it is one-to-one, the cosine inverse function only gives us one value, but we know that the period nature of the cosine function suggests that there are infinitely many solutions to an equation like this.

Since the period of the cosine function is $2\pi$ units, we can find another solution by adding any integer-multiple of the solution we found above. Thus, $t \approx 1.16 + 2k\pi,\ \ k \in \mathbb{Z}$, represents infinitely many solutions to the given equation…but it doesn't represent all of the solutions; see Figure 7 below.

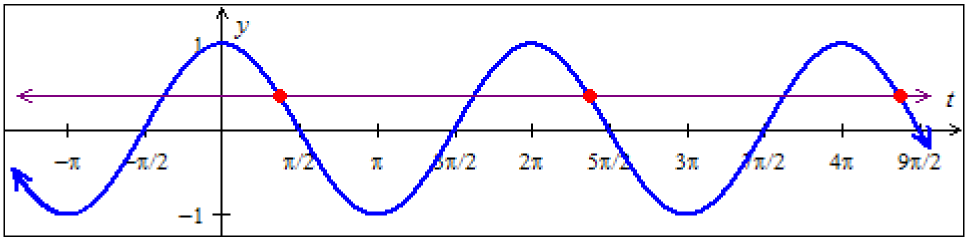


**Figure 7:** Graph of $y = \cos(t)$ and the line $y = 0.4$. The red dots represent points with horizontal coordinates of the form $t \approx 1.16 + 2k\pi,\ \ k \in \mathbb{Z}$. The other instances where the blue and purple graphs intersect are solutions to the equation $\cos(t) = 0.4$ but they are NOT represented by $t \approx 1.16 + 2k\pi,\ \ k \in \mathbb{Z}$.

It should be clear after studying Figure 1 that we are missing solutions and that one of the solutions we are missing is on the left side of the $y$-axis just as close to $y$-axis as our original solution, $t \approx 1.16$. (Recall the identity $\cos(t) = \cos(-t)$ that we noticed in Chapter 3.) It should be clear that this solution is $t \approx -1.16$, so we can represent the rest of the solutions with $t \approx -1.16 + 2k\pi$. Thus, the complete solution to $\cos(t) = 0.4$ is

$$t \approx 1.16 + 2k\pi \quad \text{or} \quad t \approx -1.16 + 2k\pi \quad \text{for all } k \in \mathbb{Z}$$

Now let's define the inverse tangent function. Recall that the tangent function is one-to-one on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; since the period of tangent is $\pi$ units, this interval represents a complete period of tangent. In order to construct the inverse tangent function, we restrict the tangent function to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

**DEFINITION:** The **inverse tangent function**, denoted $y = \tan^{-1}(t)$, is defined by the following:

$$\text{If } -\tfrac{\pi}{2} < y < \tfrac{\pi}{2} \text{ and } \tan(y) = t \text{, then } y = \tan^{-1}(t).$$

By construction, the range of $y = \tan^{-1}(t)$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and the domain is the same as the range of the tangent function: $\mathbb{R}$. Note that the inverse tangent function is often called the **arctangent function** and denoted $y = \arctan(t)$.

**EXAMPLE 3:** **a.** Evaluate $\sin^{-1}\left(-\frac{1}{2}\right)$.

**b.** Evaluate $\cos^{-1}(0)$

**c.** Evaluate $\tan^{-1}(1)$

SOLUTION:

**a.** To evaluate $\sin^{-1}\left(-\frac{1}{2}\right)$, we need to find a value, $p$, such that $-\frac{\pi}{2} \le p \le \frac{\pi}{2}$ and $\sin(p) = -\frac{1}{2}$. Our experience tells us that $p = -\frac{\pi}{6}$. Thus, $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

**b.** To evaluate $\cos^{-1}(0)$, we need to find a value, $p$, such that $0 \le p \le \pi$ and $\cos(p) = 0$. Our experience tells us that $p = \frac{\pi}{2}$. Thus, $\cos^{-1}(0) = \frac{\pi}{2}$.

**c.** To evaluate $\tan^{-1}(1)$, we need to find a value, $p$, such that $-\frac{\pi}{2} < p < \frac{\pi}{2}$ and $\tan(p) = 1$. Our experience tells us that $p = \frac{\pi}{4}$. Thus, $\tan^{-1}(1) = \frac{\pi}{4}$.

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**EXAMPLE 4:** **a.** Evaluate $\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$.

**b.** Evaluate $\cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$

**c.** Evaluate $\tan\left(\tan^{-1}\left(-\sqrt{3}\right)\right)$

SOLUTION:

**a.** To evaluate $\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$, we need to first evaluate find $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$, so we need to find a value, $p$, such that $-\frac{\pi}{2} \le p \le \frac{\pi}{2}$ and $\sin(p) = \frac{\sqrt{3}}{2}$. Our experience tells us that $p = \frac{\pi}{3}$. Thus, $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$. Now we can evaluate $\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$:

$$\begin{aligned}\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right) &= \sin\left(\frac{\pi}{3}\right)\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

**b.** To evaluate $\cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$, we need to first evaluate find $\cos^{-1}\left(\frac{1}{2}\right)$, so we need to find a value, $p$, such that $0 \le p \le \pi$ and $\cos(p) = \frac{1}{2}$. Our experience tells us that $p = \frac{\pi}{3}$. Thus, $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$. Now we can evaluate $\cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$:

$$\begin{aligned}\cos\left(\cos^{-1}\left(\tfrac{1}{2}\right)\right) &= \cos\left(\tfrac{\pi}{3}\right)\\ &= \tfrac{1}{2}\end{aligned}$$

**c.** To evaluate $\tan\left(\tan^{-1}\left(-\sqrt{3}\right)\right)$, we need to first evaluate find $\tan^{-1}\left(-\sqrt{3}\right)$, so we need to find a value, $p$, such that $-\frac{\pi}{2} < p < \frac{\pi}{2}$ and $\tan(p) = -\sqrt{3}$. Our experience tells us that $p = -\frac{\pi}{3}$. Thus, $\tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$. Now we can evaluate $\tan\left(\tan^{-1}\left(-\sqrt{3}\right)\right)$:

$$\begin{aligned}\tan\left(\tan^{-1}\left(-\sqrt{3}\right)\right) &= \tan\left(-\tfrac{\pi}{3}\right)\\ &= -\sqrt{3}\end{aligned}$$

Notice that the answers to all three parts of this example are exactly what we should have expected the answers to be since inverse functions "undo each other." (We should have studied inverse functions in our previous course-work.) But we have to be careful since the inverse sine, cosine, and tangent functions are NOT the inverses of the complete sine, cosine, and tangent functions. The next example should help explain why we need to be careful with the inverse trigonometric functions.

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**EXAMPLE 5: a.** Evaluate $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$.

**b.** Evaluate $\cos^{-1}\left(\cos(2\pi)\right)$

**c.** Evaluate $\tan^{-1}\left(\tan\left(\frac{5\pi}{4}\right)\right)$

SOLUTION:

**a.** Based on what we noticed in the last example and what we know about how inverse functions "undo each other," we might assume that $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ is equal to $\frac{2\pi}{3}$, but this is NOT true. (Notice that it can't possibly be true since the answer to this question is an output for the inverse sine function and $\frac{2\pi}{3}$ isn't in the range of this function!)

$$\begin{aligned}\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) &= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) && \text{since } \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}.\\ &= \frac{\pi}{3} && \text{since } \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \text{ and } -\frac{\pi}{2} \le \frac{\pi}{3} \le \frac{\pi}{2}.\end{aligned}$$

So $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ is equal to $\frac{\pi}{3}$, not $\frac{2\pi}{3}$, since $\frac{2\pi}{3}$ isn't in the range of $y = \sin^{-1}(t)$.

**b.** Since inverse functions "undo each other," we might assume that $\cos^{-1}\left(\cos(2\pi)\right)$ is equal to $2\pi$, but this is NOT true. (Notice that it can't possibly be true since the answer to this question is an output for the inverse cosine function and $2\pi$ isn't in the range of this function!)

$$\begin{aligned}\cos^{-1}\left(\cos(2\pi)\right) &= \cos^{-1}(1) && \text{since } \cos(2\pi) = 1.\\ &= 0 && \text{since } \cos(0) = 1 \text{ and } 0 \le 0 \le \pi.\end{aligned}$$

So $\cos^{-1}\left(\cos(2\pi)\right)$ is equal to $0$, not $2\pi$, since $2\pi$ isn't in the range of $y = \cos^{-1}(t)$.

**c.** Since inverse functions "undo each other," we might assume that $\tan^{-1}\left(\tan\left(\frac{5\pi}{4}\right)\right)$ is equal to $\frac{5\pi}{4}$, but this is NOT true. (Notice that it can't possibly be true since the answer to this question is an output for the inverse tangent function and $\frac{5\pi}{4}$ isn't in the range of this function!)

$$\begin{aligned}\tan^{-1}\left(\tan\left(\frac{5\pi}{4}\right)\right) &= \tan^{-1}(1) && \text{since } \tan\left(\frac{5\pi}{4}\right) = 1.\\ &= \frac{\pi}{4} && \text{since } \tan\left(\frac{\pi}{4}\right) = 1 \text{ and } -\frac{\pi}{2} < \frac{\pi}{4} < \frac{\pi}{2}.\end{aligned}$$

So $\tan^{-1}\left(\tan\left(\frac{5\pi}{4}\right)\right)$ is equal to $\frac{\pi}{4}$, not $\frac{5\pi}{4}$, since $\frac{5\pi}{4}$ isn't in the range of $y = \tan^{-1}(t)$.

**EXAMPLE 6:** Solve the equations below:

**a.** $2\cos(t) = -\sqrt{3}$

**b.** $13\sin(x) = 6$

**c.** $\sqrt{3}\tan(x) = 1$

SOLUTION:

**a.**

$$2\cos(t) = -\sqrt{3}$$

$$\Rightarrow \quad \cos(t) = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \cos^{-1}\big(\cos(t)\big) = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) \qquad \text{For the next step, note that } \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}.$$

$$\Rightarrow t = \frac{5\pi}{6} + 2k\pi \quad \text{or} \quad t = -\frac{5\pi}{6} + 2k\pi \quad \text{for all } k \in \mathbb{Z}$$

See the solution to Example 2, above, to review how to find *all* of the solutions to an equation involving cosine.

**b.**

$$13\sin(x) = 6$$

$$\Rightarrow \quad \sin(x) = \frac{6}{13}$$

$$\Rightarrow \quad \sin^{-1}\big(\sin(x)\big) = \sin^{-1}\left(\frac{6}{13}\right)$$

$$\Rightarrow x = \sin^{-1}\left(\frac{6}{13}\right) + 2k\pi \quad \text{or} \quad x = \pi - \sin^{-1}\left(\frac{6}{13}\right) + 2k\pi \quad \text{for all } k \in \mathbb{Z}$$

See the solution to parts **a** and **b** of the Example 1, above, to review how to find *all* of the solutions to an equation involving sine.

**c.**

$$\sqrt{3}\tan(x) = 1$$

$$\Rightarrow \quad \tan(x) = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \tan^{-1}\big(\tan(x)\big) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \qquad \text{For the next step, note that } \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

$$\Rightarrow \quad x = \frac{\pi}{6} + k\pi \quad \text{for all } k \in \mathbb{Z}$$

Notice that we add $k\pi$ (rather than $2k\pi$) to our solutions since, unlike sine and cosine, the period of tangent is $\pi$ units.

**EXAMPLE 7:** Solve the equations below:

**a.** $6\sin(2x) = 3\sqrt{2}$

**b.** $2\cos(3t) = -1$

**c.** $10\tan(5t) = 20$

SOLUTION:

**a.** Notice that the trigonometric function involved in the given equation is $\sin(2x)$, and recall that $\sin(2x)$ has period $\pi$ units, i.e., the values for $\sin(2x)$ repeat every $\pi$ units. This means that once we find a solution to the given equation we'll be able to add to it any integer multiple of $\pi$ and obtain another solution. Thus, we should expect the phrase "$k\pi$ for all $k \in \mathbb{Z}$" to be involved in our solutions. You'll see in the work below that we add $2k\pi$ to our solutions after applying the inverse sine function since the period of the sine function is $2\pi$ units. In the last step, we finish solving for $x$ and obtain the desired period-shift of $k\pi$ units.

$$6\sin(2x) = 3\sqrt{2}$$

$$\Rightarrow \quad \sin(2x) = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \quad \sin^{-1}(\sin(2x)) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

For the next step, note that $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$.

$$\Rightarrow 2x = \frac{\pi}{4} + 2k\pi \quad \text{or} \quad 2x = \pi - \frac{\pi}{4} + 2k\pi \quad \text{for all } k \in \mathbb{Z}$$

$$\Rightarrow \quad x = \frac{1}{2}\left(\frac{\pi}{4} + 2k\pi\right) \quad \text{or} \quad x = \frac{1}{2}\left(\frac{3\pi}{4} + 2k\pi\right) \quad \text{for all } k \in \mathbb{Z}$$

$$\Rightarrow \quad x = \frac{\pi}{8} + k\pi \quad \text{or} \quad x = \frac{3\pi}{8} + k\pi \quad \text{for all } k \in \mathbb{Z}$$

**b.** Notice that the trigonometric function involved in the given equation is $\cos(3t)$, and recall that $\cos(3t)$ has period $\frac{2\pi}{3}$ units, i.e., the values for $\cos(3t)$ repeat every $\frac{2\pi}{3}$ units. This means that once we find a solution to the given equation we'll be able to add to it any integer multiple of $\frac{2\pi}{3}$ and obtain another solution. Thus, we should expect the phrase "$\frac{2k\pi}{3}$ for all $k \in \mathbb{Z}$" to be involved in our solutions. You'll see in the work below that we add $2k\pi$ to our solutions after applying the inverse cosine function since the period of the cosine function is $2\pi$ units. In the last step, we finish solving for $t$ and obtain the desired period-shift of $\frac{2k\pi}{3}$ units.

$$2\cos(3t) = -1$$

$$\Rightarrow \qquad \cos(3t) = -\tfrac{1}{2}$$

$$\Rightarrow \qquad \cos^{-1}\left(\cos(3t)\right) = \cos^{-1}\left(-\tfrac{1}{2}\right) \qquad \text{For the next step, note that } \cos^{-1}\left(-\tfrac{1}{2}\right) = \tfrac{2\pi}{3}.$$

$$\Rightarrow \; 3t = \tfrac{2\pi}{3} + 2k\pi \qquad \text{or} \qquad 3t = -\tfrac{2\pi}{3} + 2k\pi \quad \text{for all } k \in \mathbb{Z}$$

$$\Rightarrow \quad t = \tfrac{1}{3}\left(\tfrac{2\pi}{3} + 2k\pi\right) \quad \text{or} \quad t = \tfrac{1}{3}\left(-\tfrac{2\pi}{3} + 2k\pi\right) \quad \text{for all } k \in \mathbb{Z}$$

$$\Rightarrow \quad t = \tfrac{2\pi}{9} + \tfrac{2k\pi}{3} \qquad \text{or} \qquad t = -\tfrac{2\pi}{9} + \tfrac{2k\pi}{3} \quad \text{for all } k \in \mathbb{Z}$$

**c.** Notice that the trigonometric function involved in the given equation is $\tan(5t)$, and recall that $\tan(5t)$ has period $\frac{\pi}{5}$ units, i.e., the values for $\tan(5t)$ repeat every $\frac{\pi}{5}$ units. This means that once we find a solution to the given equation we'll be able to add to it any integer multiple of $\frac{\pi}{5}$ and obtain another solution. Thus, we should expect the phrase "$\frac{k\pi}{5}$ for all $k \in \mathbb{Z}$ to be involved in our solutions. You'll see in the work below that we add $k\pi$ to our solutions after applying the inverse tangent function since the period of the tangent function is $\pi$ units. In the last step, we finish solving for $t$ and obtain the desired period-shift of $\frac{k\pi}{5}$ units.

$$10\tan(5t) = 20$$

$$\Rightarrow \qquad \tan(5t) = 2$$

$$\Rightarrow \quad \tan^{-1}\left(\tan(5t)\right) = \tan^{-1}(2)$$

$$\Rightarrow \qquad 5t = \tan^{-1}(2) + k\pi \quad \text{for all } k \in \mathbb{Z}$$

$$\Rightarrow \qquad t = \tfrac{1}{5}\tan^{-1}(2) + \tfrac{k\pi}{5} \quad \text{for all } k \in \mathbb{Z}$$

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**EXAMPLE 8:** Solve the equation $3\cos(2x) - 2 = 0$ for $-\pi \le x \le \pi$.

SOLUTION:

$$3\cos(2x) - 2 = 0$$

$$\Rightarrow \qquad \cos(2x) = \tfrac{2}{3}$$

$$\Rightarrow \qquad \cos^{-1}\left(\cos(2x)\right) = \cos^{-1}\left(\tfrac{2}{3}\right)$$

$$\Rightarrow 2x = \cos^{-1}\left(\tfrac{2}{3}\right) + 2k\pi \quad \text{or} \quad 2x = -\cos^{-1}\left(\tfrac{2}{3}\right) + 2k\pi \quad \text{for all } k \in \mathbb{Z}$$

$$\Rightarrow \quad x = \tfrac{1}{2}\left(\cos^{-1}\left(\tfrac{2}{3}\right) + 2k\pi\right) \quad \text{or} \quad x = \tfrac{1}{2}\left(-\cos^{-1}\left(\tfrac{2}{3}\right) + 2k\pi\right) \quad \text{for all } k \in \mathbb{Z}$$

$$\Rightarrow \quad x = \tfrac{1}{2}\cos^{-1}\left(\tfrac{2}{3}\right) + k\pi \quad \text{or} \quad x = -\tfrac{1}{2}\cos^{-1}\left(\tfrac{2}{3}\right) + k\pi \quad \text{for all } k \in \mathbb{Z}$$

Notice that we were asked to find the solutions in the interval $-\pi \le x \le \pi$, so we need to find which of the infinitely many solutions we've found are in the interval. It might help if we approximate the values we found above:

$$x = \tfrac{1}{2}\cos^{-1}\left(\tfrac{2}{3}\right) + k\pi \approx 0.42 + k\pi$$

or

$$x = -\tfrac{1}{2}\cos^{-1}\left(\tfrac{2}{3}\right) + k\pi \approx -0.42 + k\pi \quad \text{for all } k \in \mathbb{Z}$$

We know that $\pi \approx 3.14$ so we need to find values that satisfy the equation as above and are between $-3.14$ and $3.14$.

$k = -1$: $x \approx 0.42 + (-1)\cdot\pi \approx -2.72$ or $x \approx -0.42 + (-1)\cdot\pi \approx -3.56$

Only $-2.72$ is in the desired interval.

$k = 0$: $x \approx 0.42 + 0\cdot\pi \approx 0.42$ or $x \approx -0.42 + 0\cdot\pi \approx -0.42$

Both of these values are in the desired interval.

$k = 1$: $x \approx 0.42 + 1\cdot\pi \approx 3.56$ or $x \approx -0.42 + 1\cdot\pi \approx 2.72$

Only $2.72$ is in the desired interval.

$k = 2$: $x \approx 0.42 + 2\cdot\pi \approx 6.7$ or $x \approx -0.42 + 2\cdot\pi \approx 5.86$

Neither of these values is in the desired interval.

We could try more $k$-values, but we can tell from the work we've done thus far that no other $k$-values will give us solutions that are in the interval $-\pi \le x \le \pi$. Thus, the solution to $3\cos(2x) - 2 = 0$ for $-\pi \le x \le \pi$ is

$$x \approx -2.72, \; x \approx -0.42, \; x \approx 0.42, \text{ or } x \approx 2.72$$