

Section I: The Trigonometric Functions



Chapter 6: Graphing Sinusoidal Functions



DEFINITION: A **sinusoidal function** is function f of the form

$$f(t) = A \sin(B(t - h)) + k, \quad A, B, h, k \in \mathbb{R}.$$

NOTE 1: Recall from Chapter 3 that $\cos(t) = \sin\left(t + \frac{\pi}{2}\right)$ (i.e., since the cosine function is a transformation of the sine function). Therefore, we can write any function of the form $f(t) = A \cos(B(t - h)) + k$ in the form given in the definition of a sinusoidal function so functions of the form $f(t) = A \cos(B(t - h)) + k$ are sinusoidal functions.

NOTE 2: Based what we know about graph transformations (which are studied in the previous course), we should recognize that sinusoidal functions are transformations of the function $y = \sin(t)$. Below is a summary of what is studied in the previous course about graph transformations:

SUMMARY OF GRAPH TRANSFORMATIONS

Suppose that f and g are functions such that $g(t) = A \cdot f(B(t - h)) + k$ and $A, B, h, k \in \mathbb{R}$. In order to transform the graph of the function f into the graph of g ...

- 1st:** horizontally stretch/compress the graph of f by a factor of $\frac{1}{|B|}$ and, if $B < 0$, reflect it about the y -axis.
- 2nd:** shift the graph horizontally h units (shift right if h is positive and left if h is negative).
- 3rd:** vertically stretch/compress the graph by a factor of $|A|$ and, if $A < 0$, reflect it about the t -axis.
- 4th:** shift the graph vertically k units (shift up if k is positive and down if k is negative).

(The order in which these transformations are performed matters.)

Examples 1 – 4 (below) will provide a review of the graph transformations as well as an investigation of the affect of the constants A , B , h , and k on the period, midline, amplitude, and horizontal shift of a sinusoidal function. You may want to follow along by graphing the functions on your graphing calculator. Don't forget to change the **mode** of the calculator to the **radian** setting under the heading **angle**.



EXAMPLE 1: Describe how we can transform the graph of $f(t) = \sin(t)$ into the graph of $g(t) = 2\sin(t) - 3$. State the period, midline, and amplitude of $y = g(x)$.

SOLUTION:

Our goal is to use Examples 1 – 4 to determine how the constants A , B , h , and k affect the period, midline, amplitude, and horizontal shift of a sinusoidal function so let's start by observing what the values of A , B , h , and k are in $g(t) = 2\sin(t) - 3$. It should be clear that function g is a sinusoidal function of the form $y = A\sin(B(t - h)) + k$ where $A = 2$, $B = 1$, $h = 0$, and $k = -3$.

After inspecting the rules for the functions f and g , we should notice that we could construct the function $g(t) = 2\sin(t) - 3$ by multiplying the outputs of the function $f(t) = \sin(t)$ by 2 and then subtracting 3 from the result. We can express this algebraically with the equation below:

$$g(t) = 2f(t) - 3$$

Based on what we know about graph transformations, we can conclude that we can obtain graph of g by starting with the graph of f and first stretching it vertically by a factor of 2 and then shifting it down 3 units. Since $f(t) = \sin(t)$ has amplitude 1 unit, if we stretch it vertically by a factor of 2 then we'll double the amplitude, so we should expect that the amplitude of g to be 2 units. Also, since $f(t) = \sin(t)$ has midline $y = 0$, when we shift it down 3 units to draw the graph of g , the resulting midline will be $y = -3$. (Note that since graphing g required no horizontal transformations of $f(t) = \sin(t)$, the graph of g must have the same period as the graph of $f(t) = \sin(t)$: 2π units.) Let's summarize what we've learned about $g(t) = 2\sin(t) - 3$:

period: 2π units

midline: $y = -3$

amplitude: 2 units

horizontal shift: 0 units

The graphs of $f(t) = \sin(t)$ and $g(t) = 2\sin(t) - 3$ are given in Figure 1 below.

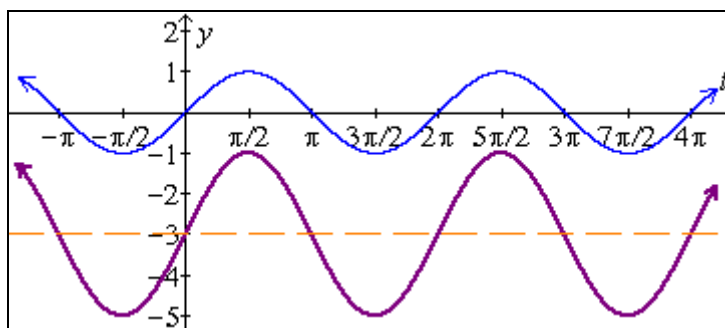


Figure 1: The graphs of $f(t) = \sin(t)$ and $g(t) = 2\sin(t) - 3$.



EXAMPLE 2: Describe how we can transform the graph of $f(t) = \sin(t)$ into the graph of $n(t) = \sin\left(t + \frac{\pi}{4}\right)$; state the period, midline, amplitude, and horizontal shift of $y = n(x)$.

SOLUTION:

Notice that the function n is a sinusoidal function of the form $y = A\sin(B(t - h)) + k$ where $A = 1$, $B = 1$, $h = -\frac{\pi}{4}$, and $k = 0$.

After inspecting the rules for the functions f and n , it should be clear that we can write n in terms of f as follows: $n(t) = f\left(t + \frac{\pi}{4}\right)$. Based on what we know about graph transformations, we can conclude that we can obtain graph of n by starting with the graph of f and shifting it left $\frac{\pi}{4}$ units. Since a horizontal shift won't affect the period, midline, or amplitude, we should expect that the period, midline, and amplitude of $n(t) = \sin\left(t + \frac{\pi}{4}\right)$ are the same as $f(t) = \sin(t)$:

period: 2π units

midline: $y = 0$

amplitude: 1 unit

horizontal shift: $-\frac{\pi}{4}$ units

The graphs of $f(t) = \sin(t)$ and $n(t) = \sin\left(t + \frac{\pi}{4}\right)$ are given in Figure 2.

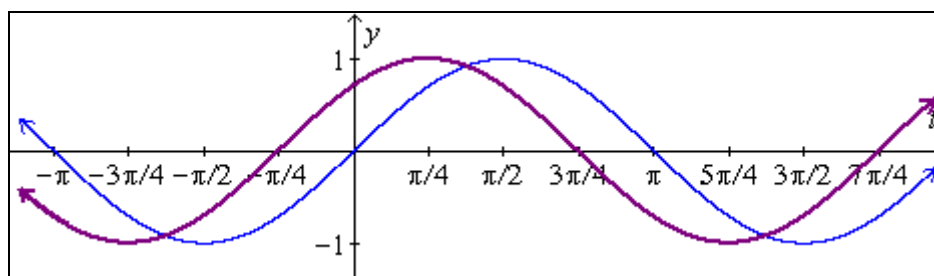


Figure 2: The graphs of $f(t) = \sin(t)$ (blue) and $n(t) = \sin\left(t + \frac{\pi}{4}\right)$ (purple).



EXAMPLE 3: Describe how we can transform the graph of $p(t) = \cos(t)$ into the graph of $q(t) = -\cos(2t)$ and find the period, midline, amplitude, and horizontal shift of $y = q(x)$.

SOLUTION:

Notice that the function q is a sinusoidal function of the form $y = A\cos(B(t - h)) + k$ where $A = -1$, $B = 2$, $h = 0$, and $k = 0$.

After inspecting the rules for the functions p and q , it should be clear that we can write q in terms of p as follows: $q(t) = -p(2t)$. Based on what we know about graph transformations, we can conclude that we can obtain graph of q by starting with the graph of p and first stretching it horizontally by a factor of $\frac{1}{2}$ (i.e., compressing the graph by a factor of 2) and then reflecting it about the t -axis. Since $p(t) = \cos(t)$ has period 2π units, if we compress the graph by a factor of 2 then the period will be shrunk to π units. Since we aren't stretching the graph of p vertically, we should expect that the amplitude of q is the same as the amplitude of p : 1 unit. Also, since we aren't shifting the graph of p vertically, we should expect that the midline of q is the same as the midline of p : $y = 0$. Let's summarize what we've learned about $q(t) = -\cos(2t)$:

period: π units

midline: $y = 0$

amplitude: 1 unit

horizontal shift: 0 units

The graphs of $p(t) = \cos(t)$ and $q(t) = -\cos(2t)$ are given in Figure 3.

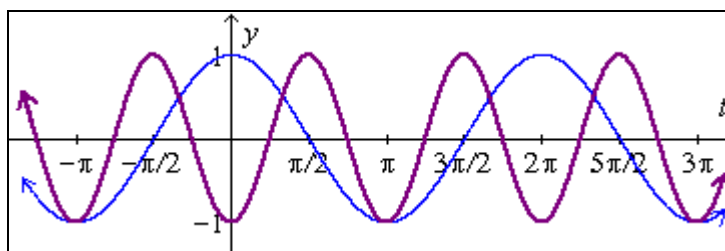


Figure 3: The graphs of $p(t) = \cos(t)$ (blue) and $q(t) = -\cos(2t)$ (purple).

Notice that the graph of $q(t) = -\cos(2t)$ completes **two** periods in the interval $[0, 2\pi]$. In general, the number **B** in a sinusoidal function of the form $y = A\sin(B(t - h)) + k$ or $y = A\cos(B(t - h)) + k$ represents the number of periods (or “cycles”) that the function completes on an interval of length 2π . This number B is called the **angular frequency** of a sinusoidal function.

When we use sinusoidal functions to represent real-life situations, we often take the input variable to be a unit of *time*. Suppose that in the function $q(t) = -\cos(2t)$, t represents seconds. Since the input of the cosine function *must* be radians, the units of $B = 2$ must be “radians per second”. This way,

$$2 \frac{\text{radians}}{\text{second}} \cdot t \text{ seconds} = 2t \text{ radians},$$

which has the appropriate units for the input of the cosine function. So if t represents seconds, the **angular frequency** of $q(t) = -\cos(2t)$ is “2 radians per second”.

Another way to obtain the unit of the angular frequency is to use what we noticed above: the number **2** in $q(t) = -\cos(\mathbf{2}t)$ represents the number of cycles that the function completes on an interval of length 2π . Since a cycle is equivalent to a complete rotation around a circle, or 2π radians, two cycles is equivalent to 4π radians. If the input variable t represents seconds, then the angular frequency is

$$\frac{4\pi \text{ radians}}{2\pi \text{ seconds}} = 2 \text{ rad/sec.}$$



EXAMPLE 4: Describe how we can transform the graph of $p(t) = \cos(t)$ into the graph $w(t) = 3p\left(\frac{1}{2}\left(t - \frac{\pi}{3}\right)\right) + 5$. State the period, midline, amplitude, and horizontal shift of $y = w(x)$.

SOLUTION:

Notice that the function w is a sinusoidal function of the form $y = A\cos(B(t - h)) + k$ where $A = 3$, $B = \frac{1}{2}$, $h = \frac{\pi}{3}$, and $k = 5$.

After inspecting the rules for the functions p and w , it should be clear that we can write w in terms of p as follows: $w(t) = 3p\left(\frac{1}{2}\left(t - \frac{\pi}{3}\right)\right) + 5$. Based on what we know about graph transformations, we can conclude that we can obtain graph of q by starting with the graph of p and first stretching it horizontally by a factor of 2, then shifting it right $\frac{\pi}{3}$ units, then stretching it vertically by a factor 3, and finally shifting it up 5 units. Since $p(t) = \cos(t)$ has period 2π units, if we stretch the graph by a factor of 2 then the period will be stretched to 4π units. Similarly, if we stretch the graph of $p(t) = \cos(t)$ vertically by a factor of 3 then we'll triple the amplitude, so we should expect the amplitude of w to be 3 units. Also, since $p(t) = \cos(t)$ has midline $y = 0$, when we shift it up 5 units to draw the graph of w , the resulting midline will be $y = 5$. Since we are shifting the graph

right $\frac{\pi}{3}$ units, the horizontal shift is $\frac{\pi}{3}$ units. Let's summarize what we've learned about $y = w(x)$:

period: 4π units

midline: $y = 5$

amplitude: 3 units

horizontal shift: $\frac{\pi}{3}$ units

The graphs of $p(t) = \cos(t)$ and $w(t) = 3p\left(\frac{1}{2}\left(t - \frac{\pi}{3}\right)\right) + 5$ are given in Figure 4.

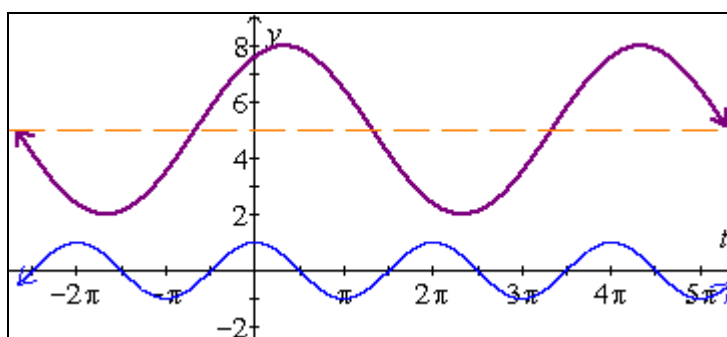


Figure 4: The graphs of $p(t) = \cos(t)$ (blue) and $w(t) = 3\cos\left(\frac{1}{2}\left(t - \frac{\pi}{3}\right)\right) + 5$ (purple).

Notice that the graph of $w(t) = 3\cos\left(\frac{1}{2}\left(t - \frac{\pi}{3}\right)\right) + 5$ completes **one-half** of a period (or “cycle”) in the interval $[0, 2\pi]$. If we let the input variable, t , represent seconds, then $w(t) = 3p\left(\frac{1}{2}\left(t - \frac{\pi}{3}\right)\right) + 5$ completes one-half of a cycle every 2π seconds. Since one-half of a cycle is equivalent to half of a rotation around a circle, or π radians, then the **angular frequency** of the function w is

$$\frac{\pi \text{ radians}}{2\pi \text{ seconds}} = \frac{1}{2} \text{ rad/sec.}$$

Based on what we learned in the examples above, we can summarize the affect of the constants A , B , h , and k on the period, midline, amplitude, and horizontal shift of functions of the form $y = A\sin(B(t - h)) + k$ and $y = A\cos(B(t - h)) + k$.

SUMMARY: Graphs of Sinusoidal Functions

The graphs of the sinusoidal functions

$$y = A\sin(B(t - h)) + k \quad \text{and} \quad y = A\cos(B(t - h)) + k$$

(where $A, B, h, k \in \mathbb{R}$) have the following properties:

period: $\frac{2\pi}{|B|}$ units

midline: $y = k$

amplitude: $|A|$ units

horizontal shift: h units

angular frequency: B radians per unit of t



EXAMPLE 5: Sketch a graph of $f(t) = 2\sin\left(\pi t - \frac{\pi}{4}\right) - 3$.

SOLUTION:

In order to use what we've just studied about functions of the form $y = A\sin(B(t - h)) + k$, we need to write the given function in this form, i.e., we need to factor π (which is playing the role of " B ") out of the input expression " $\pi t - \frac{\pi}{4}$ ":

$$\begin{aligned} f(t) &= 2\sin\left(\pi t - \frac{\pi}{4}\right) - 3 \\ &= 2\sin\left(\pi\left(t - \frac{1}{4}\right)\right) - 3 \end{aligned}$$

It should be clear that $f(t) = 2\sin\left(\pi\left(t - \frac{1}{4}\right)\right) - 3$ is a sinusoidal function of the form $y = A\sin(B(t - h)) + k$ where $A = 2$, $B = \pi$, $h = \frac{1}{4}$, and $k = -3$. Using what we found above, we can find the period, midline, amplitude, and horizontal shift of $y = f(t)$:

period: $\frac{2\pi}{|B|} = \frac{2\pi}{|\pi|} = 2$ units

midline: $y = -3$

amplitude: $|2| = 2$ unit

horizontal shift: $\frac{1}{4}$ of a unit

We can use this information to sketch a graph of $f(t) = 2\sin\left(\pi\left(t - \frac{1}{4}\right)\right) - 3$; see Figure 5 below. (Note that the horizontal shift tells us where to “start” our usual sine wave.)

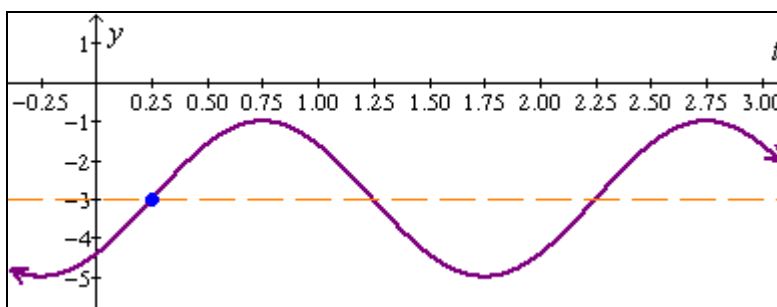


Figure 5: The graph of $f(t) = 2\sin\left(\pi\left(t - \frac{1}{4}\right)\right) - 3$. The blue point represents where we “start” our sine wave since the horizontal shift is $\frac{1}{4}$ of a unit.

Note that, according to what we discussed in Examples 3 and 4, if we let t represent seconds then we could state that the **angular frequency** of $f(t) = 2\sin\left(\pi\left(t - \frac{1}{4}\right)\right) - 3$ is π radians per second. Since π radians represents one-half of a rotation around a circle, the angular frequency “ π radians per second t ,” is equivalent to one-half of a cycle per second. Notice that our graph in Figure 5 shows a function that completes one-half of a period in one unit of t !