Section I: The Trigonometric Functions

Chapter 5: Intro to the Trigonometric Functions, Part 3

Recall that the sine and cosine function represent the coordinates of points in the circumference of a unit circle. In Chapter 4, we found the sine and cosine values for 30°, 45°, and 60° (i.e., for $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$) by finding the coordinates of the points on the circumference of the unit circle specified by these angles. The points we found were all in Quadrant I but, since a circle is symmetric about both the x and y axes, we can reflect these points about the coordinate axes to determine the coordinates of corresponding points in the other quadrants. This means that we can use the sine and cosine values of $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$ to find the sine and cosine values of corresponding angles in the other quadrants.

Because of the symmetry of a circle, we can take a point in Quadrant I and reflect it about the x-axis, the y-axis, and about both axes in order to obtain corresponding points, one in each of the three other quadrants; the absolute value of the coordinates of all four of these points is the same, i.e., they only differ by their signs. In Figure 1, we've plotted the point (a, b) specified by angle θ in Quadrant I along with the corresponding points in the other quadrants.

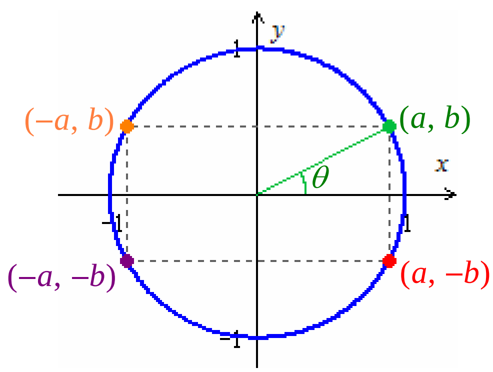

Figure 1

Notice that if you construct a segment between the origin and each of these four points, then the acute angle between this segment and the x-axis is the same angle, θ; see Figure 2.

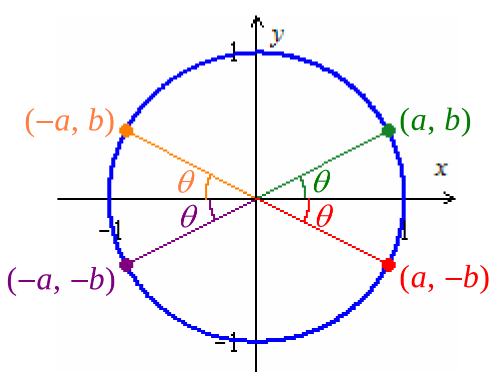

Figure 2

Although all four of the points in Figure 2 are specified by a different angle, all four of the angles share the same *reference angle*, θ.

DEFINITION: The **reference angle** for an angle is the acute (i.e., smaller than 90°) angle between the terminal side of the angle and the x-axis.

EXAMPLE 1: a. Find the reference angle for 150°.

b. Find the reference angle for $\frac{5\pi}{4}$.

SOLUTION:

a. The reference angle is for 150° is 30°; see Figure 3.

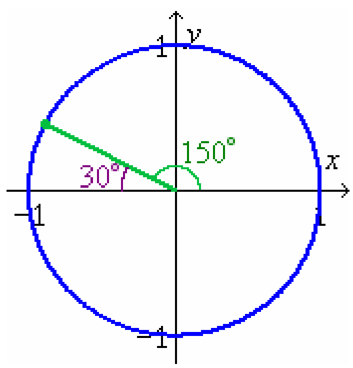

Figure 3

b. The reference angle for $\frac{5\pi}{4}$ is $\frac{\pi}{4}$; see Figure 4.

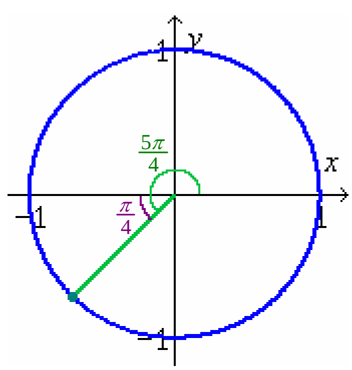

Figure 4

Now let's discuss how we can use reference angles to determine the sine and cosine of any integer multiple of $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$.

In Chapter 4, we determined the sine and cosine values of $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$ which gave us the exact the coordinates of the points on the unit circle specified by these angles; see Figure 5.

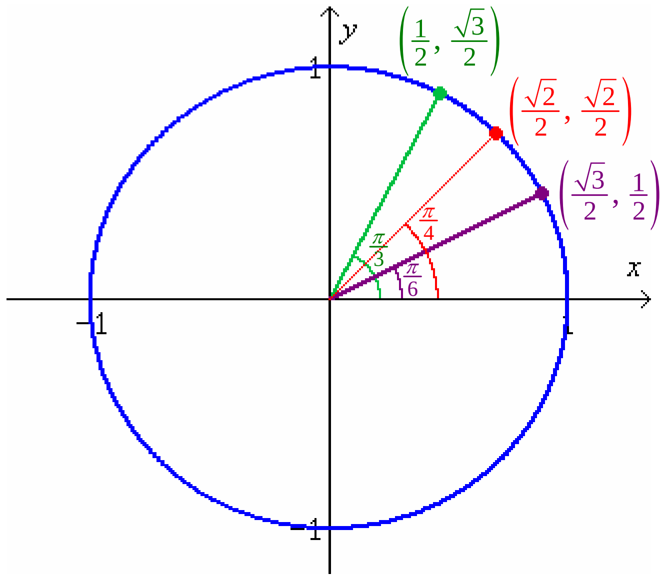

Figure 5

We can use the information in Figure 5 to find the sine and cosine of any angle that has $\frac{\pi}{6}$, $\frac{\pi}{4}$, or $\frac{\pi}{3}$ as its reference angle.

First let's focus on angles with reference angle $\frac{\pi}{4}$.

Notice that both the horizontal and vertical coordinates of the point on the unit circle specified by $\frac{\pi}{4}$ are both $\frac{\sqrt{2}}{2}$. Of course, this means that $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, so whenever we are working with $\frac{\pi}{4}$, we should remember that we are going to use the number $\frac{\sqrt{2}}{2}$. Let's consider an example:

EXAMPLE 2a: Find $\sin\left(\frac{5\pi}{4}\right)$ and $\cos\left(\frac{5\pi}{4}\right)$.

SOLUTION:

As we observed in Example 1, the reference angle for $\frac{5\pi}{4}$ is $\frac{\pi}{4}$ so we know that the absolute value of $\sin\left(\frac{5\pi}{4}\right)$ will be the same as $\sin\left(\frac{\pi}{4}\right)$ and the absolute value of $\cos\left(\frac{5\pi}{4}\right)$ will be the same as $\cos\left(\frac{\pi}{4}\right)$ but, since $\frac{5\pi}{4}$ is in the third quadrant, both its sine and cosine values will be *negative*. We know that $\frac{\pi}{4}$ has a sine and cosine value of $\frac{\sqrt{2}}{2}$, so we can conclude that

$$\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} \quad \text{and} \quad \cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}.$$

Figure 6 shows this information communicated as the coordinate of the point specified by $\frac{5\pi}{4}$ on the circumference of a unit circle.

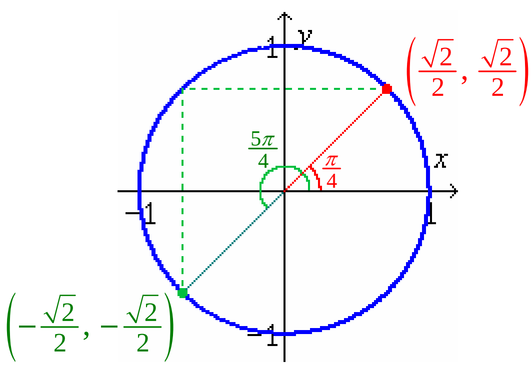

Figure 6

EXAMPLE 2b: Use Example 2a to find $\tan\left(\frac{5\pi}{4}\right)$, $\cot\left(\frac{5\pi}{4}\right)$, $\sec\left(\frac{5\pi}{4}\right)$, $\csc\left(\frac{5\pi}{4}\right)$.

SOLUTION:

$$\tan\left(\frac{5\pi}{4}\right) = \frac{\sin\left(\frac{5\pi}{4}\right)}{\cos\left(\frac{5\pi}{4}\right)} = \frac{-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = 1$$

$$\cot\left(\frac{5\pi}{4}\right) = \frac{1}{\tan\left(\frac{5\pi}{4}\right)} = \frac{1}{1} = 1$$

$$\sec\left(\frac{5\pi}{4}\right) = \frac{1}{\cos\left(\frac{5\pi}{4}\right)} = \frac{1}{-\frac{\sqrt{2}}{2}} = -\frac{2}{\sqrt{2}}$$

$$\csc\left(\frac{5\pi}{4}\right) = \frac{1}{\sin\left(\frac{5\pi}{4}\right)} = \frac{1}{-\frac{\sqrt{2}}{2}} = -\frac{2}{\sqrt{2}}$$

Now let's focus on angles with a reference angle of either $\frac{\pi}{6}$ or $\frac{\pi}{3}$. Recall from Figure 5 that $\frac{\pi}{6}$ specifies the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ on the unit circle and that specifies $\frac{\pi}{3}$ the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ on the unit circle. Thus, the horizontal and vertical coordinates of the points specified by $\frac{\pi}{6}$ or $\frac{\pi}{3}$ are either $\frac{1}{2}$ or $\frac{\sqrt{3}}{2}$ (these are the *only* options), so whenever we are working with $\frac{\pi}{6}$ or $\frac{\pi}{3}$, we should remember that we are going to use either $\frac{1}{2}$ or $\frac{\sqrt{3}}{2}$. But we need a way to decide which of these two numbers we need to use.

Notice that $\frac{1}{2} < \frac{\sqrt{3}}{2}$ and that $\frac{\pi}{6} < \frac{\pi}{3}$, and observe that when the horizontal coordinate is larger than the vertical coordinate, i.e., if the ordered pair is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, then the point is close to the x-axis and the angle that specifies the point is a small angle, i.e., $\frac{\pi}{6}$. Similarly, observe that when the horizontal coordinate is smaller than the vertical coordinate, i.e., if the ordered pair is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, then the point is further above the x-axis and the angle that specifies the point is a large angle, i.e., $\frac{\pi}{3}$. So, when the angle is smaller there hasn't been much rotation so the horizontal coordinate is larger and the vertical coordinate is smaller, but when the angle is larger, there has been substantial rotation so the vertical coordinate is larger and the horizontal coordinate is smaller. (Spend some time with this paragraph until it makes sense.)

Let's use this way of thinking to evaluate a few expressions.

EXAMPLE 3: Find $\sin\left(\frac{\pi}{3}\right)$ and $\cos\left(\frac{\pi}{6}\right)$.

- To find $\sin\left(\frac{\pi}{3}\right)$, first take note that the function is sine, so it's a *vertical* coordinate that we're looking for. Next, consider the angle, $\frac{\pi}{3}$. This is the angle that, along with $\frac{\pi}{6}$, has sine and cosine values of $\frac{1}{2}$ or $\frac{\sqrt{3}}{2}$, so we know that we have to chose one of these for our sine value. Since $\frac{\pi}{3}$ is larger than $\frac{\pi}{6}$, it specifies a point on the unit circle with a larger vertical coordinate so the sine value must be the larger of our two choices so we can conclude that $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

- To find $\cos\left(\frac{\pi}{6}\right)$, first take note that the function is cosine, so it's a *horizontal* coordinate that we're looking for. Next, consider the angle, $\frac{\pi}{6}$, and note that, along with $\frac{\pi}{3}$, it has

sine and cosine values of $\frac{1}{2}$ or $\frac{\sqrt{3}}{2}$, so we know that we have to chose one of these for our cosine value. Since $\frac{\pi}{6}$ is smaller than $\frac{\pi}{3}$, it specifies a point on the unit circle with a larger horizontal coordinate and smaller vertical coordinate, so the cosine value must be the larger of our two choices so we can conclude that $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

Now we can use what we know about $\frac{\pi}{6}$ and $\frac{\pi}{3}$ to find the sine and cosine of angles in other quadrants that have $\frac{\pi}{6}$ or $\frac{\pi}{3}$ as their reference angle.

EXAMPLE 4: Find $\sin\left(\frac{5\pi}{3}\right)$.

SOLUTION:

To find $\sin\left(\frac{5\pi}{3}\right)$, first take note that the function is sine, so it's a *vertical* coordinate that we're looking for. Next, consider the angle, $\frac{5\pi}{3}$; it's in Quadrant IV and vertical coordinates are negative in Quadrant IV, so we know that our sine value is negative. Since $\frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$, we see that the reference angle for $\frac{5\pi}{3}$ is $\frac{\pi}{3}$, so the absolute value of our sine value must be either $\frac{1}{2}$ or $\frac{\sqrt{3}}{2}$. (In the discussion above, we noticed that these are our only choices when we're working with $\frac{\pi}{3}$.) Since $\frac{\pi}{3}$ is larger than $\frac{\pi}{6}$ we know that $\frac{\pi}{3}$ specifies a point on the unit circle with a larger vertical coordinate, so we know that we'll need to use the larger of $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ for our sine value, so we can conclude that $\sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$; see Figure 7.

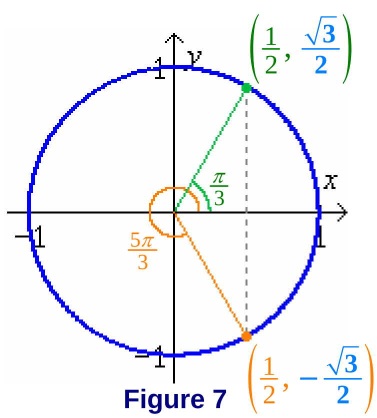

Figure 7

EXAMPLE 5: Find $\cos\left(\frac{5\pi}{6}\right)$.

SOLUTION:

To find $\cos\left(\frac{5\pi}{6}\right)$, first take note that the function is cosine, so it's a *horizontal* coordinate that we're looking for. Next, consider the angle, $\frac{5\pi}{6}$; it's in Quadrant II and horizontal coordinates are negative in Quadrant II, so we know that our cosine value is negative. Since $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$, we see that the reference angle for $\frac{5\pi}{6}$ is $\frac{\pi}{6}$, so the absolute value of our cosine value must be either $\frac{1}{2}$ or $\frac{\sqrt{3}}{2}$. Since $\frac{\pi}{6}$ is smaller than $\frac{\pi}{3}$, we know that $\frac{\pi}{6}$ specifies a point on the unit circle with a larger horizontal coordinate, so we know that we'll need to use the larger of $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ for our cosine value, so we can conclude that $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$; see Figure 8.

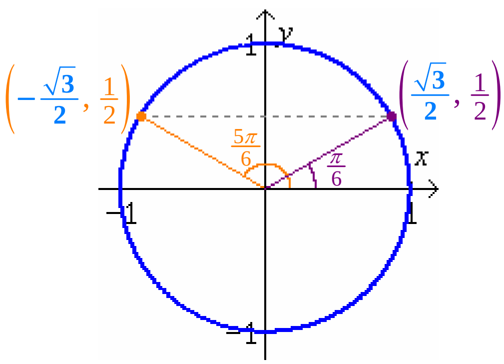

Figure 8

EXAMPLE 6: Find $\cos\left(\frac{4\pi}{3}\right)$.

SOLUTION:

To find $\cos\left(\frac{4\pi}{3}\right)$, first take note that the function is cosine, so it's a *horizontal* coordinate that we're looking for. Next, consider the angle, $\frac{4\pi}{3}$; it's in Quadrant III and horizontal coordinates are negative in Quadrant III, so we know that our cosine value is negative. Since $\frac{4\pi}{3} = \pi + \frac{\pi}{3}$, we see that the reference angle for $\frac{4\pi}{3}$ is $\frac{\pi}{3}$, so the absolute value of our cosine value must be either $\frac{1}{2}$ or $\frac{\sqrt{3}}{2}$. Since $\frac{\pi}{3}$ is larger than $\frac{\pi}{6}$, we know that $\frac{\pi}{3}$

specifies a point on the unit circle with a smaller horizontal coordinate, so we know that we'll need to use the smaller of $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ for our cosine value, so we can conclude that $\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$; see Figure 9.

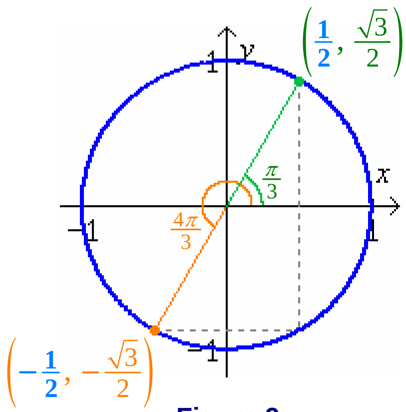

Figure 9


EXAMPLE 7a: Find $\cos(150^\circ)$ and $\sin(150^\circ)$.

SOLUTION:

As shown in Figure 10, the reference angle is for 150° is 30° so the sine and cosine values for 150° are the same as the sine and cosine values of 30° except, since 150° is in Quadrant II, the cosine value is negative. Thus,

$$\begin{aligned}\cos(150^\circ) &= -\cos(30^\circ)\\ &= -\tfrac{\sqrt{3}}{2}\end{aligned} \qquad \text{and} \qquad \begin{aligned}\sin(150^\circ) &= \sin(30^\circ)\\ &= \tfrac{1}{2}\end{aligned}$$

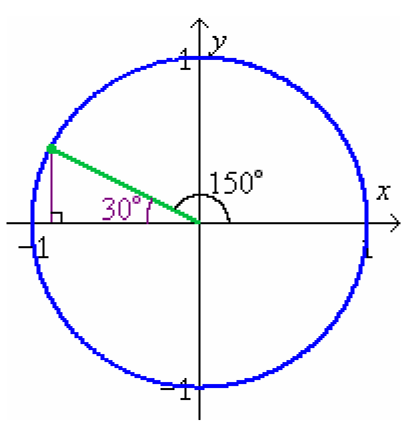

Figure 10

EXAMPLE 7b: Use Example 7a to find $\tan(150^\circ)$, $\cot(150^\circ)$, $\sec(150^\circ)$, $\csc(150^\circ)$.

SOLUTION:

$\tan(150^\circ) = \dfrac{\sin(150^\circ)}{\cos(150^\circ)}$	$\cot(150^\circ) = \dfrac{1}{\tan(150^\circ)}$	$\sec(150^\circ) = \dfrac{1}{\cos(150^\circ)}$	$\csc(150^\circ) = \dfrac{1}{\sin(150^\circ)}$
$= \dfrac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$	$= \dfrac{1}{-\frac{1}{\sqrt{3}}}$	$= \dfrac{1}{-\frac{\sqrt{3}}{2}}$	$= \dfrac{1}{\frac{1}{2}}$
$= -\dfrac{1}{\sqrt{3}}$	$= -\sqrt{3}$	$= -\dfrac{2}{\sqrt{3}}$	$= 2$

EXAMPLE 8a: Find $\cos(405^\circ)$ and $\sin(405^\circ)$.

SOLUTION:

Since $405^\circ = 360^\circ + 45^\circ$, 405° is co-terminal with 45°; see Figure 11.

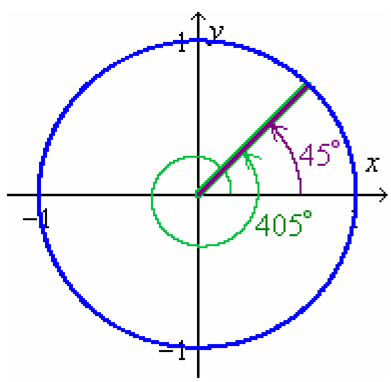

Figure 11

Therefore, the reference angle for 405° is 45°, and the sine and cosine values for 405° are the same as the sine and cosine of 45°. Thus,

$$\cos(405^\circ) = \cos(45^\circ) = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin(405^\circ) = \sin(45^\circ) = \frac{\sqrt{2}}{2}$$

EXAMPLE 8b: Use Example 8a to find $\tan(405^\circ)$, $\cot(405^\circ)$, $\sec(405^\circ)$, $\csc(405^\circ)$.

SOLUTION:

$$\begin{aligned}\tan(405^\circ) &= \frac{\sin(405^\circ)}{\cos(405^\circ)} \\ &= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \\ &= 1\end{aligned}$$

$$\begin{aligned}\cot(405^\circ) &= \frac{1}{\tan(405^\circ)} \\ &= \frac{1}{1} \\ &= 1\end{aligned}$$

$$\begin{aligned}\sec(405^\circ) &= \frac{1}{\cos(405^\circ)} \\ &= \frac{1}{\frac{\sqrt{2}}{2}} \\ &= \frac{2}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\csc(405^\circ) &= \frac{1}{\sin(405^\circ)} \\ &= \frac{1}{\frac{\sqrt{2}}{2}} \\ &= \frac{2}{\sqrt{2}}\end{aligned}$$

EXAMPLE 9: A circle with a radius of 6 units is given in Figure 12. The point Q is specified by the angle $\frac{2\pi}{3}$. Use the sine and cosine function to express the exact coordinates of point Q.

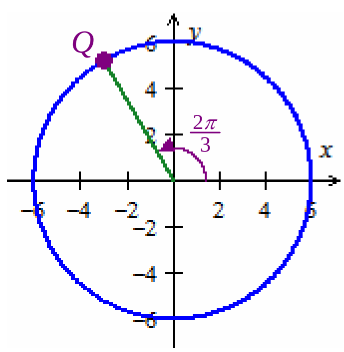

Figure 12

SOLUTION:

The point Q is specified by $\frac{2\pi}{3}$ on the circumference of a circle of radius 6 units. Thus,

$$\begin{aligned}Q &= \left(6\cos\left(\tfrac{2\pi}{3}\right),\ 6\sin\left(\tfrac{2\pi}{3}\right)\right) \\ &= \left(6\cdot\left(-\tfrac{1}{2}\right),\ 6\cdot\left(\tfrac{\sqrt{3}}{2}\right)\right) \\ &= \left(-3,\ 3\sqrt{3}\right)\end{aligned}$$