Section I: Periodic Functions and Trigonometry

Chapter 1: The Unit Circle, Radians, and Arc-Length

In this chapter we will study a few definitions and concepts that we'll use throughout the course.

DEFINITION: A **unit circle** is a circle with a radius, r, of 1 unit.

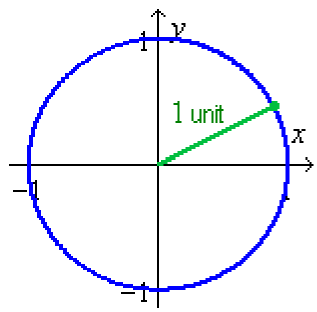

Figure 1: A Unit Circle

Now let's take note of some conventions and terminology that we will use when discussing angles within circles, like angle θ in Figure 2.

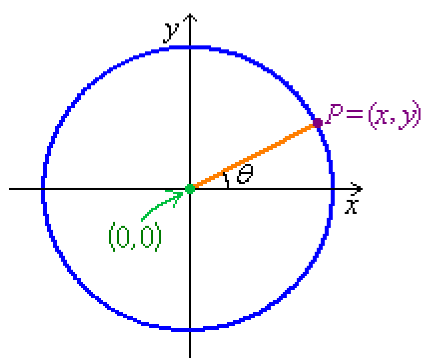

Figure 2

- The angle θ is measured **counterclockwise from the positive x-axis**.
- The segment between the origin, $(0, 0)$, and the point P is the ***terminal side of angle θ***.
- Two angles with the same terminal side are said to be ***co-terminal angles***.
- The point P on the circumference of the circle is said to be ***specified by the angle θ***.

- Angle θ corresponds with a portion of the circumference of the circle called the **arc spanned by θ**; see Figure 3.

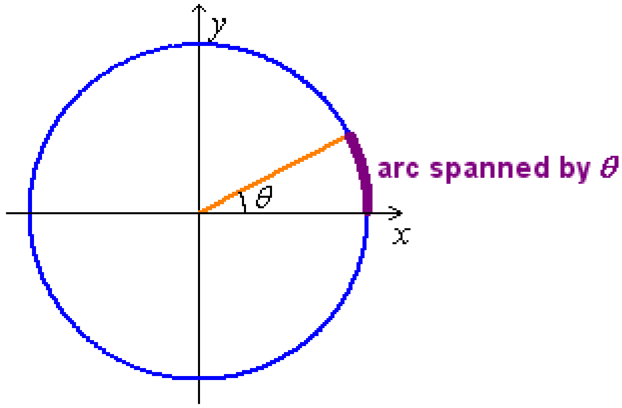

Figure 3

Thus far in your mathematics careers you have probably measured angles in **degrees**. Three hundred and sixty degrees (360°) represents a complete rotation around a circle, so 1° corresponds to $1/360^{\text{th}}$ of a complete rotation.

As noted above, angles are measured counterclockwise from the positive x-axis; consequently, negative angles are measured *clockwise* from the positive x-axis; see Figure 4.

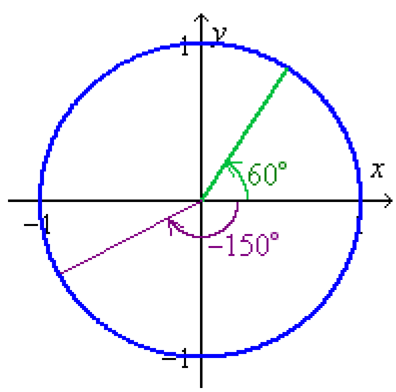

Figure 4

We mentioned above that *co-terminal angles* share the same terminal side. Since 360° represents a complete rotation about the circle, if we add any integer multiple of 360° to an angle θ_1, we'll obtain an angle co-terminal to θ_1. In other words, the angles

$$\theta_1 \quad \text{and} \quad \theta_2 = \theta_1 + 360^\circ \cdot k \quad \text{where} \quad k \in \mathbb{Z}$$

are co-terminal. For example, the angles 45° and $45^\circ + 360^\circ = 405^\circ$ are co-terminal; see Figure 5.

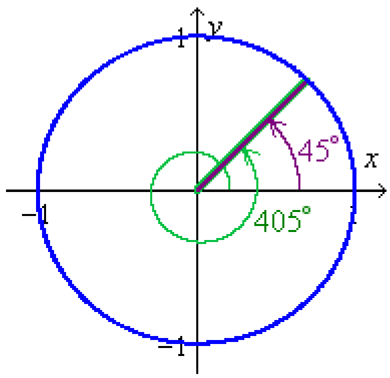

Figure 5: The angles 45° and 405° are co-terminal.

Traditionally, the coordinate plane is divided into ***four quadrants***; see Figure 6. We will often use the names of these quadrants to describe the location of the terminal side of different angles.

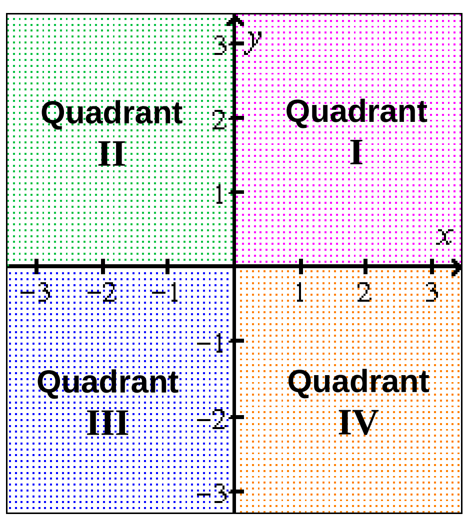

Figure 6

For example, consider the angles given in Figure 4: the angle 60° is in Quadrant I while -150° is in Quadrant III.

Instead of using degrees to measure angles, we can use ***radians***.

DEFINITION: The **radian** measure of an angle is the ratio of the length of the arc on the circumference of the circle spanned by the angle and the radius of the circle; see Figure 7. Since a radian is a ratio of two lengths, the length-units cancel; thus, radians are considered a ***unit-less*** measure.

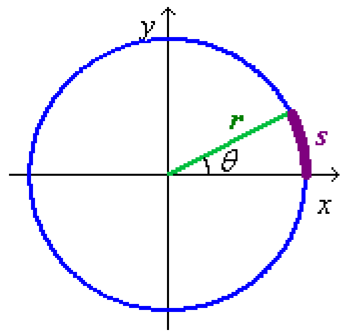

Figure 7: The angle θ measures $\frac{s}{r}$ radians.

NOTE: An alternative yet equivalent definition is that an angle that measures 1 **radian** is defined to be an angle at the center of a unit circle (measured counterclockwise) which spans an arc of length 1 unit on the circumference of the circle; see Figure 8.

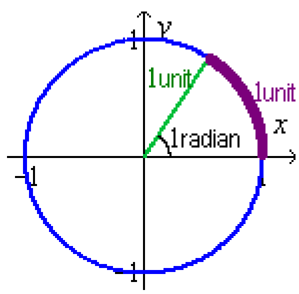

Figure 8.

Since, *on a unit circle*, the radian measure of an angle and the arc-length spanned by an angle are the *same* value, in order to find the radian measure of a complete rotation around a circle (i.e., 360°), we need to find the arc-length of an entire unit circle. Of course, the arc-length of an entire circle is the *circumference* of the circle; recall that that circumference, c, of a circle is given by the formula $c = 2\pi r$ where r is the radius of the circle. Thus, the circumference of the unit circle (i.e., arc-length of the complete unit circle) is $c = 2\pi \cdot 1 = 2\pi$ units. Therefore, the radian measure of a complete rotation about a circle (i.e., 360°) is equivalent to 2π radians. We can state this symbolically as follows:

$$360^\circ = 2\pi \text{ radians}$$

The equation above implies that the following two ratios equal 1; we can use these ratios to convert from degrees to radians, and vise versa:

$$\frac{2\pi \text{ rad}}{360^\circ} = \frac{360^\circ}{2\pi \text{ rad}} = 1.$$

EXAMPLE: **a.** How many degrees are 8 radians?

b. How many radians are 8 degrees?

SOLUTION:

a. In order to convert 8 radians into degrees, we can multiply 8 radians by $\frac{360^\circ}{2\pi \text{ rad}}$. (Since this equals 1, multiplying by it won't change the value of our angle-measure.)

$$\begin{aligned} 8 \cancel{\text{ rad}} \cdot \frac{360^\circ}{2\pi \cancel{\text{ rad}}} &= \frac{8 \cdot 360^\circ}{2\pi} \\ &= \frac{1440^\circ}{\pi} \\ &\approx 458.37^\circ \end{aligned}$$

Thus, 8 radians is about 458.37°.

b. In order to convert 8 degrees into radians, we can multiply 8° by $\frac{2\pi \text{ rad}}{360^\circ}$. (Since this equals 1, multiplying by it won't change the value of our angle-measure.)

$$\begin{aligned} 8\cancel{^\circ} \cdot \frac{2\pi \text{ rad}}{360\cancel{^\circ}} &= \frac{16\pi}{360} \text{ rad} \\ &= \frac{2\pi}{45} \text{ rad} \\ &\approx 0.14 \text{ rad} \end{aligned}$$

Thus, 8° is about 0.14 radians.

EXAMPLE: **a.** Convert 1 radian into degrees.

b. Convert 90° into radians.

SOLUTION:

a. In order to convert 1 radian into degrees, we can multiply 1 radian by $\frac{360^\circ}{2\pi \text{ rad}}$.

$$1 \cancel{\text{rad}} \cdot \frac{360^\circ}{2\pi \cancel{\text{rad}}} = \frac{360^\circ}{2\pi}$$
$$= \frac{180^\circ}{\pi}$$
$$\approx 57.3^\circ$$

Thus, 1 radian is about $57.3°$.

b. In order to convert 90° into radians, we can multiply 90° by $\frac{2\pi \text{ rad}}{360^\circ}$.

$$90\cancel{^\circ} \cdot \frac{2\pi \text{ rad}}{360\cancel{^\circ}} = \frac{180\pi}{360} \text{ rad}$$
$$= \frac{\pi}{2} \text{ rad}$$

Thus, 90° is equivalent to $\frac{\pi}{2}$ radians.

EXAMPLE: Complete the table below:

θ (degrees)	0°	30°	45°	60°	90°	180°	270°	360°
θ (radians)								

SOLUTION:

θ (degrees)	0°	30°	45°	60°	90°	180°	270°	360°
θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Recall the definition of *radian*: the radian measure of an angle is the ratio of the length of the arc on the circumference of the circle spanned by the angle and the radius of the circle. Applying this fact to the circle in Figure 9 if θ is measured in radians, then

$$\theta = \frac{\text{arc-length}}{\text{radius}} = \frac{s}{r}$$

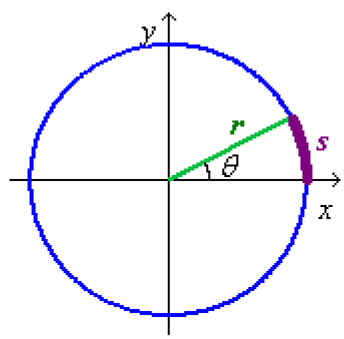

Figure 9: Circle of radius r with an angle θ spanning an arc-length s.

By solving the equation $\theta = \dfrac{s}{r}$ for s, we obtain the following definition:

DEFINITION: The **arc-length**, s, spanned in a circle of radius r by an angle θ radians is given by

$$s = r|\theta|.$$

Note we need the absolute value of θ so that we obtain a positive arc-length if θ is negative. (Lengths are always positive!) Also, note that this formula only works if θ is measured in radians.)

EXAMPLE: **a.** What is the arc-length spanned by an angle of 2 radians on a circle of radius 5 inches?

b. What is the arc-length spanned by an angle of 30° on a circle of radius 20 meters?

SOLUTION:

a. To find the arc-length, we can use the formula $s = r|\theta|$.

$$\begin{aligned} s &= r|\theta| \\ &= 5 \cdot 2 \\ &= 10 \end{aligned}$$

Thus, the arc-length spanned by an angle of 2 radians on a circle of radius 5 inches is 10 inches.

b. Before we can use the formula $s = r|\theta|$, we need to convert the angle into radians. In order to convert 30° into radians, we can multiply 30° by $\frac{2\pi \text{ rad}}{360^\circ}$ (which equals 1). (Of course we could use the table we created earlier in this chapter, but we will go ahead and show the computation here.)

$$\begin{aligned} 30\cancel{^\circ} \cdot \frac{2\pi \text{ rad}}{360\cancel{^\circ}} &= \frac{60\pi}{360} \text{ rad} \\ &= \frac{\pi}{6} \text{ rad} \end{aligned}$$

Thus, 30° is equivalent to $\frac{\pi}{6}$ radians. Now we can find the desired arc-length:

$$\begin{aligned} s &= r|\theta| \\ &= 20 \cdot \frac{\pi}{6} \\ &= \frac{10\pi}{3} \end{aligned}$$

Thus, the arc-length spanned by an angle of 30° on a circle of radius 20 meters is $\frac{10\pi}{3}$ meters.