

## Section I: Periodic Functions and Trigonometry

### Chapter 0: Sets and Numbers



**DEFINITION:** A **set** is a collection of objects specified in a manner that enables one to determine if a given object is or is not in the set.

In other words, a set is a well-defined collection of objects.



**EXAMPLE:** Which of the following represent a set?

- a. The students registered for MTH 112 at PCC this quarter.
- b. The good students registered for MTH 112 at PCC this quarter.

**SOLUTION:**

- a. This represents a set since it is “well defined”: We all know what it means to be registered for a class.
- b. This does NOT represent a set since it is not well defined: There are many different understandings of what it means to be a good student (get an A or pass the class or attend class or avoid falling asleep in class).



**EXAMPLE:** Which of the following represent a set?

- a. All of the really big numbers.
- b. All the whole numbers between 3 and 10.

**SOLUTION:**

- a. It should be obvious why this does NOT represent a set. (What does it mean to be a “big number”?)
- b. This represents a set. We can represent sets like **b** in **roster notation** by using “curly brackets”.

“All of the whole numbers between 3 and 10” =  $\{4, 5, 6, 7, 8, 9\}$

**Roster Notation** involves listing the elements in a set within *curly brackets*: “{ }”.



**DEFINITION:** An object in a set is called an **element** of the set. ( symbol: “ $\in$ ” )



**EXAMPLE:** 5 is an element of the set  $\{4, 5, 6, 7, 8, 9\}$ . We can express this symbolically:

$$5 \in \{4, 5, 6, 7, 8, 9\}$$



**DEFINITION:** Two sets are considered **equal** if they have the same elements.

We used this definition earlier when we wrote:

$$\text{“All of the whole numbers between 3 and 10”} = \{4, 5, 6, 7, 8, 9\}.$$



**DEFINITION:** A set  $S$  is a **subset** of a set  $T$ , denoted  $S \subseteq T$ , if all elements of  $S$  are also elements of  $T$ .

If  $S$  and  $T$  are sets and  $S = T$ , then  $S \subseteq T$ . Sometimes it is useful to consider a subset  $S$  of a set  $T$  that is not equal to  $T$ . In such a case, we write  $S \subset T$  and say that  $S$  is a **proper subset** of  $T$ .



**EXAMPLE:**  $\{4, 7, 8\}$  is a subset of the set  $\{4, 5, 6, 7, 8, 9\}$ .

We can express this fact symbolically by  $\{4, 7, 8\} \subseteq \{4, 5, 6, 7, 8, 9\}$ .

Since these two sets are not equal,  $\{4, 7, 8\}$  is a *proper* subset of  $\{4, 5, 6, 7, 8, 9\}$ , so we can write

$$\{4, 7, 8\} \subset \{4, 5, 6, 7, 8, 9\}.$$



**DEFINITION:** The **empty set**, denoted  $\emptyset$ , is the set with no elements.

$$\emptyset = \{ \quad \}$$

There are NO elements in  $\emptyset$ .

The empty set is a subset of all sets. Note that  $0 \neq \emptyset$ .



**DEFINITION:** The **union** of two sets  $A$  and  $B$ , denoted  $A \cup B$ , is the set containing all of the elements in either  $A$  or  $B$  (or both  $A$  and  $B$ ).



**EXAMPLE:** Consider the sets  $\{4, 7, 8\}$ ,  $\{0, 2, 4, 6, 8\}$ , and  $\{1, 3, 5, 7\}$ . Then...

- a.  $\{4, 7, 8\} \cup \{1, 3, 5, 7\} = \{1, 3, 4, 5, 7, 8\}$
- b.  $\{4, 7, 8\} \cup \{0, 2, 4, 6, 8\} = \{0, 2, 4, 6, 7, 8\}$
- c.  $\{0, 2, 4, 6, 8\} \cup \{1, 3, 5, 7\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$



**DEFINITION:** The **intersection** of two sets  $A$  and  $B$ , denoted  $A \cap B$ , is the set containing all of the elements in both  $A$  and  $B$ .



**EXAMPLE:** Consider the sets  $\{4, 7, 8\}$ ,  $\{0, 2, 4, 6, 8\}$ , and  $\{1, 3, 5, 7\}$ . Then...

- a.  $\{4, 7, 8\} \cap \{0, 2, 4, 6, 8\} = \{4, 8\}$
- b.  $\{4, 7, 8\} \cap \{1, 3, 5, 7\} = \{7\}$
- c.  $\{0, 2, 4, 6, 8\} \cap \{1, 3, 5, 7\} = \emptyset$  These sets have no elements in common, so their intersection is the empty set.



**EXAMPLE:** All of the whole numbers (positive and negative) form a set. This set is called the **integers**, and is represented by the symbol  $\mathbb{Z}$ . We can express the set of integers in roster notation:

$$\mathbb{Z} = \{ \dots, -3, -2, -1, 0, 1, 2, 3, \dots \}$$

Note that  $\mathbb{Z}$  is used to represent the integers because the German word for "number" is "zahlen."

Now that we have the integers, we can represent sets like "All of the whole numbers between 3 and 10" using **set-builder notation**:

**SET-BUILDER NOTATION:**

$$\text{"All the whole numbers between 3 and 10"} = \{x \mid x \in \mathbb{Z} \text{ and } 3 < x < 10\}$$

↑ This vertical line means "such that"

Armed with set-builder notation, we can define important **sets of numbers**:



**DEFINITIONS:** The set of **natural numbers**:  $\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$

The set of **integers**:  $\mathbb{Z} = \{ \dots, -3, -2, -1, 0, 1, 2, 3, \dots \}$

The set of **rational numbers** (i.e., the set of fractions):

$$\mathbb{Q} = \left\{ x \mid x = \frac{p}{q} \text{ and } p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$$

The set of **real numbers**:  $\mathbb{R}$  (All the numbers on the number line.)

The set of **complex numbers**:

$$\mathbb{C} = \{x \mid x = a + bi \text{ and } a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}\}$$

Note that  $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ , i.e., the set of natural numbers is a subset of the set of integers which is a subset of the set of rational numbers which is a subset of the real numbers which is a subset of the set of complex numbers.

Throughout this course, we will assume that the number-set in question is the real numbers,  $\mathbb{R}$ , unless we are specifically asked to consider an alternative set.

Since we use the real numbers so often, we have special notation for subsets of the real numbers: **interval notation**. Interval notation involves square or round brackets. Use the examples below to understand how interval notation works.



**EXAMPLE:**

$$\text{a. } \left\{ x \mid x \in \mathbb{R} \text{ and } -2 \leq x \leq 3 \right\} = [-2, 3]$$

$\uparrow$                        $\uparrow$   
 Set-builder Notation      Interval Notation

We use square brackets here since the endpoints are included

$$\text{b. } \left\{ x \mid x \in \mathbb{R} \text{ and } -2 < x < 3 \right\} = (-2, 3)$$

$\uparrow$                        $\uparrow$   
 Set-builder Notation      Interval Notation

We use round brackets here since the endpoints are NOT included.

$$\text{c. } \left\{ x \mid x \in \mathbb{R} \text{ and } -2 < x \leq 3 \right\} = (-2, 3]$$

$\uparrow$                        $\uparrow$   
 Set-builder Notation      Interval Notation

We use a round bracket on the left since  $-2$  is NOT included.

$$\text{d. } \left\{ x \mid x \in \mathbb{R} \text{ and } -2 \leq x < 3 \right\} = [-2, 3)$$

$\uparrow$                        $\uparrow$   
 Set-builder Notation      Interval Notation

We use a round bracket on the right since  $3$  is NOT included.



**EXAMPLE:** When the interval has no upper (or lower) bound, the symbol  $\infty$  (or  $-\infty$ ) is used.

$$\text{a. } \left\{ x \mid x \in \mathbb{R} \text{ and } x \leq 4 \right\} = (-\infty, 4]$$

$\uparrow$                        $\uparrow$   
 Set-builder Notation      Interval Notation

We ALWAYS use a round bracket with  $-\infty$  since it is NOT a number in the set.

$$\text{b. } \left\{ x \mid x \in \mathbb{R} \text{ and } x \geq 4 \right\} = [4, \infty)$$

$\uparrow$                        $\uparrow$   
 Set-builder Notation      Interval Notation

We ALWAYS use a round bracket with  $\infty$  since it is NOT a number in the set.



**EXAMPLE:** Simplify the following expressions.

a.  $(-4, \infty) \cup [-8, 3]$

b.  $(-4, \infty) \cup (-\infty, 2]$

c.  $(-4, \infty) \cap (-\infty, 2]$

d.  $(-4, \infty) \cap [-10, -5]$

**SOLUTION:**

a.  $(-4, \infty) \cup [-8, 3] = [-8, \infty)$

b.  $(-4, \infty) \cup (-\infty, 2] = (-\infty, \infty) = \mathbb{R}$

c.  $(-4, \infty) \cap (-\infty, 2] = (-4, 2]$

d.  $(-4, \infty) \cap [-10, -5] = \emptyset$

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