Section I: The Trigonometric Functions

Chapter 7: Right Triangle Trigonometry

As we studied in Part 1 of Chapter 3, if we put the same angle in the center of two circles of different radii, we can construct two *similar triangles*; see Figure 1.

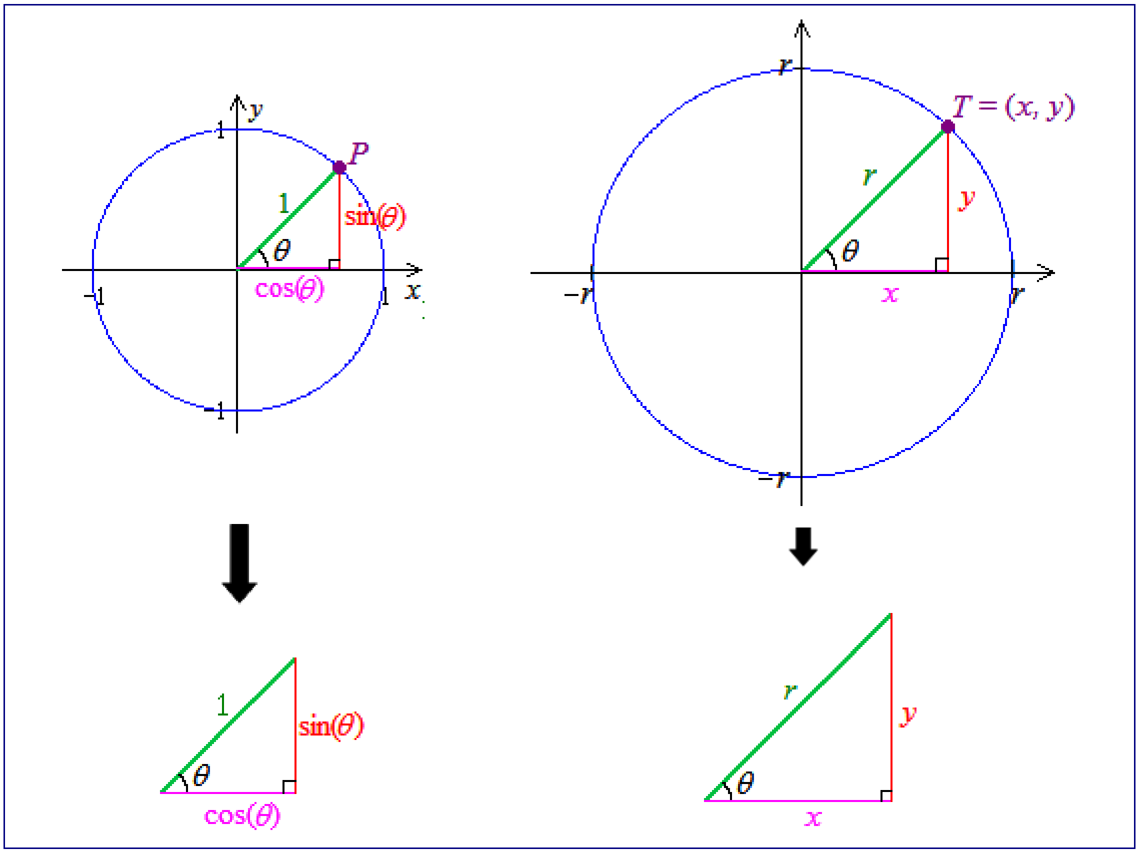

Figure 1: The angle θ in both a unit circle and in a circle of radius r, inducing similar right triangles.

We can use these similar triangles to obtain the following ratios:

$$\frac{\cos(\theta)}{1} = \frac{x}{r} \quad \text{and} \quad \frac{\sin(\theta)}{1} = \frac{y}{r}$$

Solving these ratios for $\cos(\theta)$ and $\sin(\theta)$, respectively, gives us the following:

$$\cos(\theta) = \frac{x}{r} \quad \text{and} \quad \sin(\theta) = \frac{y}{r}$$

To help us remember these ratios, it's best to imagine yourself standing at angle θ looking into the triangle. Then the side labeled "y" is on the **opposite** side of the triangle while the side labeled "x" is **adjacent** to you. We use these descriptions (as well as the fact that the side labeled "r" is the **hypotenuse** of the triangle) to refer to the sides of the triangle in Fig. 2.

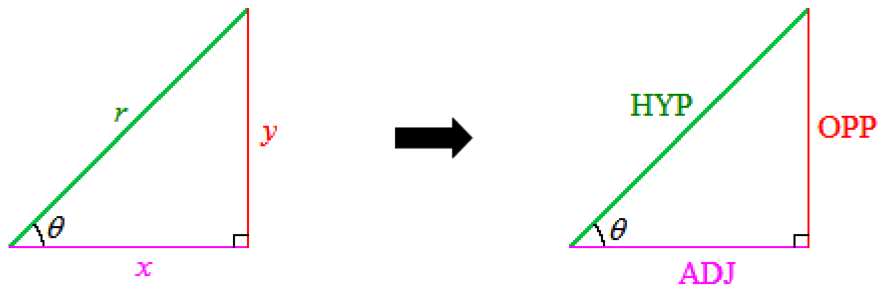

Figure 2: We use the terms **opposite** (or **OPP**), **adjacent** (or **ADJ**), and **hypotenuse** (or **HYP**) to refer to the sides of a right triangle.

DEFINITION: If θ is the angle given in the right triangles in Figure 2 (above), then

$$\sin(\theta) = \frac{y}{r} = \frac{\text{OPP}}{\text{HYP}} \quad \text{and} \quad \cos(\theta) = \frac{x}{r} = \frac{\text{ADJ}}{\text{HYP}}.$$

Consequently, the other trigonometric functions can be defined as follows:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{\text{OPP}}{\text{HYP}}}{\frac{\text{ADJ}}{\text{HYP}}} = \frac{\text{OPP}}{\text{ADJ}} \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{\frac{\text{ADJ}}{\text{HYP}}}{\frac{\text{OPP}}{\text{HYP}}} = \frac{\text{ADJ}}{\text{OPP}}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{\frac{\text{ADJ}}{\text{HYP}}} = \frac{\text{HYP}}{\text{ADJ}} \qquad \csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{\frac{\text{OPP}}{\text{HYP}}} = \frac{\text{HYP}}{\text{OPP}}$$

We can use these ratios along with the Pythagorean Theorem (see below) to learn a great deal about right triangles.

THE PYTHAGOREAN THEOREM:

If the sides of a right triangle (i.e., a triangle with a 90° angle) are labeled like the one given in Figure 3, then $\boldsymbol{a}^2 + \boldsymbol{b}^2 = \boldsymbol{c}^2$.

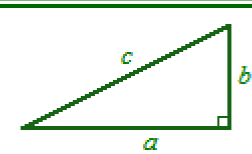

Figure 3

 **EXAMPLE 1:** Find the value for all six trigonometric functions of the angle α given in the right triangle in Figure 4. (The triangle might not be drawn to scale.)

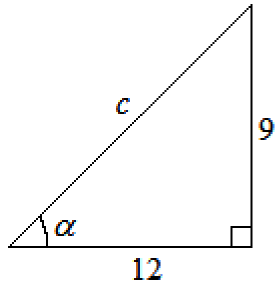

Figure 4

SOLUTION:

First we need to use the Pythagorean Theorem to find the length of the hypotenuse c.

$$
\begin{aligned}
&(12)^2 + (9)^2 = c^2 \\
\Rightarrow\ & 144 + 81 = c^2 \\
\Rightarrow\ & c^2 = 225 \\
\Rightarrow\ & c = 15
\end{aligned}
$$

We can use this value to label our triangle:

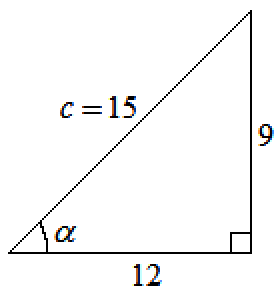

Figure 5

Thus,

$$\sin(\alpha) = \frac{\text{OPP}}{\text{HYP}} = \frac{9}{15} = \frac{3}{5} \qquad \cos(\alpha) = \frac{\text{ADJ}}{\text{HYP}} = \frac{12}{15} = \frac{4}{5}$$

$$\tan(\alpha) = \frac{\text{OPP}}{\text{ADJ}} = \frac{9}{12} = \frac{3}{4} \qquad \cot(\alpha) = \frac{\text{ADJ}}{\text{OPP}} = \frac{12}{9} = \frac{4}{3}$$

$$\sec(\alpha) = \frac{\text{HYP}}{\text{ADJ}} = \frac{15}{12} = \frac{5}{4} \qquad \csc(\alpha) = \frac{\text{HYP}}{\text{OPP}} = \frac{15}{9} = \frac{5}{3}$$

EXAMPLE 2: Find the value for all six trigonometric functions of the angle β given in the right triangle in Figure 6. (The triangle might not be drawn to scale.)

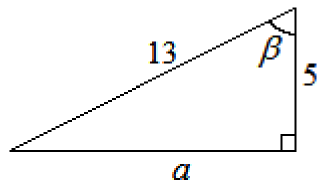

Figure 6

SOLUTION:

First we need to use the Pythagorean Theorem to find the length of the side labeled a.

$$\begin{aligned} & a^2 + (5)^2 = (13)^2 \\ \Rightarrow \quad & a^2 + 25 = 169 \\ \Rightarrow \quad & a^2 = 144 \\ \Rightarrow \quad & a = 12 \end{aligned}$$

We can use this value to label our triangle:

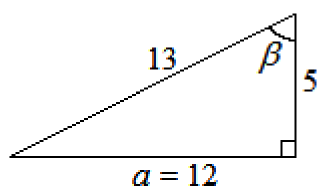

Figure 7

To determine the sine and cosine values of angle β, imagine standing at angle β and looking into the triangle. Then,

$$\sin(\alpha) = \frac{\text{OPP}}{\text{HYP}} = \frac{12}{13} \qquad \cos(\alpha) = \frac{\text{ADJ}}{\text{HYP}} = \frac{5}{13}$$

$$\tan(\alpha) = \frac{\text{OPP}}{\text{ADJ}} = \frac{12}{5} \qquad \cot(\alpha) = \frac{\text{ADJ}}{\text{OPP}} = \frac{5}{12}$$

$$\sec(\alpha) = \frac{\text{HYP}}{\text{ADJ}} = \frac{13}{5} \qquad \csc(\alpha) = \frac{\text{HYP}}{\text{OPP}} = \frac{13}{12}$$

We can use the trigonometric functions, along with the Pythagorean Theorem to "**solve a right triangle**," i.e., find the missing side-lengths and missing angle-measures for a triangle.

EXAMPLE 3: Solve the triangle in Figure 8 by finding c, α, and β. (The triangle might not be drawn to scale.)

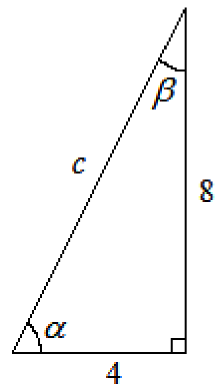

Figure 8

SOLUTION:

We can use the Pythagorean Theorem to find c.

$$\begin{aligned} & (4)^2 + (8)^2 = c^2 \\ \Rightarrow \quad & 16 + 64 = c^2 \\ \Rightarrow \quad & c^2 = 80 \\ \Rightarrow \quad & c = 4\sqrt{5} \end{aligned}$$

Now we can use the tangent function to find α. Note that we choose to use tangent, not sine or cosine, to find the α since it allows us to use the given info, rather than info that we've found. (If we made a mistake finding c, we don't want to compound that mistake but using the incorrect value to find other values.)

$$\begin{aligned} & \tan(\alpha) = \frac{8}{4} \\ \Rightarrow \quad & \tan(\alpha) = 2 \\ \Rightarrow \quad & \alpha = \tan^{-1}(2) \\ \Rightarrow \quad & \alpha \approx 63.43^\circ \end{aligned}$$

Note also that we could have just as easily found β first, instead of α. No matter which angle we find first, we can easily find the last angle by using the fact that the sum of the angles in a triangle is 180°:

$$\begin{aligned} \alpha \quad + \beta + 90^\circ &= 180^\circ \\ \Rightarrow \quad 63.43^\circ + \beta + 90^\circ &\approx 180^\circ \\ \Rightarrow \quad \beta &\approx 180^\circ - 90^\circ - 63.43^\circ \\ \Rightarrow \quad \beta &\approx 26.57^\circ \end{aligned}$$

Let's summarize our findings: $c = 4\sqrt{5}$, $\alpha \approx 63.43^\circ$, and $\beta \approx 26.57^\circ$.

EXAMPLE 4: Solve the triangle in Figure 9 by finding b, α, and β. (The triangle might not be drawn to scale.)

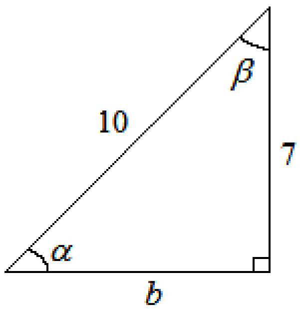

Figure 9

SOLUTION:

We can use the Pythagorean Theorem to find b.

$$\begin{aligned} 7^2 + b^2 &= 10^2 \\ \Rightarrow \quad 49 + b^2 &= 100 \\ \Rightarrow \quad b^2 &= 51 \\ \Rightarrow \quad b &= \sqrt{51} \end{aligned}$$

Now we can use the sine function to find α:

$$\begin{aligned} \sin(\alpha) &= \frac{7}{10} \\ \Rightarrow \quad \alpha &= \sin^{-1}\left(\frac{7}{10}\right) \\ \Rightarrow \quad \alpha &\approx 44.42^\circ \end{aligned}$$

Although it would be just as easy to use tangent or cosine to find α, to we choose to use sine since it allows us to use the given info, rather than info that we've found, in order to avoid the possibility of compounding our mistakes.

Now we can use the fact that the sum of the angles in a triangle is 180°:

$$\begin{aligned} \alpha \quad + \beta + 90^\circ &= 180^\circ \\ \Rightarrow \quad 44.42^\circ + \beta + 90^\circ &\approx 180^\circ \\ \Rightarrow \quad \beta &\approx 180^\circ - 90^\circ - 44.42^\circ \\ \Rightarrow \quad \beta &\approx 45.52^\circ \end{aligned}$$

Let's summarize our findings: $b = \sqrt{51}$, $\alpha \approx 44.42^\circ$, and $\beta \approx 45.52^\circ$.

EXAMPLE 5: Solve the right triangle given in Figure 10 by finding A, b, and c. (The triangle might not be drawn to scale.)

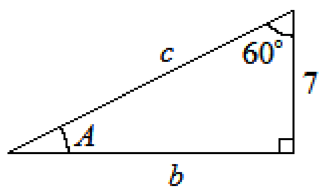

Figure 10

SOLUTION:

First, notice that angle A must measure 30° since the sum of the angles in a triangle is 180° and our triangle already has angles measuring 60° and 90°.

The only thing we know about the sides of the triangle is that the side "adjacent" to the 60° angle is 7 units long. Notice that the cosine of the 60° angle is $\frac{7}{c}$, and we can use this fact to find c:

$$\begin{aligned} \cos(60^\circ) &= \frac{7}{c} \\ \Rightarrow \quad c \cdot \cos(60^\circ) &= \frac{7}{c} \cdot c \\ \Rightarrow \quad c \cdot \frac{1}{2} &= 7 \qquad \text{(since } \cos(60^\circ) = \tfrac{1}{2}\text{)} \\ \Rightarrow \quad c &= \frac{7}{1/2} \\ \Rightarrow \quad c &= 14 \end{aligned}$$

Now that we know the length of two sides of the triangle, we could use the Pythagorean Theorem to find the length of the third side, b. Instead, we'll use the fact that, on the triangle, the sine of 60° is $\frac{b}{c}$:

$$
\begin{aligned}
& \sin(60^\circ) = \frac{b}{c} \\
\Rightarrow\quad & \sin(60^\circ) = \frac{b}{14} && \text{(since } c = 14\text{)} \\
\Rightarrow\quad & 14 \cdot \frac{\sqrt{3}}{2} = b && \text{(since } \sin(60^\circ) = \frac{\sqrt{3}}{2}\text{)} \\
\Rightarrow\quad & b = 14 \cdot \frac{\sqrt{3}}{2} \\
\Rightarrow\quad & b = 7\sqrt{3}
\end{aligned}
$$

Let's summarize our findings: $A = 30^\circ$, $b = 7\sqrt{3}$, and $c = 14$.