Section I: The Trigonometric Functions

Chapter 4: Graphing Sinusoidal Functions

DEFINITION: A **sinusoidal function** is function of the form

$$y = A\sin\big(\omega(t-h)\big)+k \quad \text{or} \quad y = A\cos\big(\omega(t-h)\big)+k,$$

where $A, \omega, h, k \in \mathbb{R}$.

Based what we know about graph transformations (which are studied in the previous course), we should recognize that a sinusoidal function is a transformation of $y=\sin(t)$ or $y=\cos(t)$. Consequently, sinusoidal functions are waves with the same curvy shape as the graphs of sine and cosine but with different periods, midlines, and/or amplitudes.

Below is a summary of what we studied about graph transformations in the previous course. We'll use this information in order to graph sinusoidal functions.

SUMMARY OF GRAPH TRANSFORMATIONS

Suppose that f and g are functions such that $g(t) = A\cdot f\big(\omega(t-h)\big)+k$ and $A, \omega, h, k \in \mathbb{R}$. In order to transform the graph of the function f into the graph of g…

1st: horizontally stretch/compress the graph of f by a factor of $\frac{1}{|\omega|}$ and, if $\omega < 0$, reflect it about the y-axis.

2nd: shift the graph horizontally h units (shift right if h is positive and left if h is negative).

3nd: vertically stretch/compress the graph by a factor of $|A|$ and, if $A<0$, reflect it about the t-axis.

4th: shift the graph vertically k units (shift up if k is positive and down if k is negative).

(The *order* in which these transformations are performed *matters*.)

Examples 1 – 4 (below) will provide a review of the graph transformations as well as an investigation of the affect of the constants A, ω, h, and k on the period, midline, amplitude, and horizontal shift of a sinusoidal function. You may want to follow along by graphing the functions on your graphing calculator. Don't forget to change the **mode** of the calculator to the **radian** setting under the heading **angle**.

EXAMPLE 1: Describe how we can transform the graph of $f(t) = \sin(t)$ into the graph of $g(t) = 2\sin(t) - 3$. State the period, midline, and amplitude of $y = g(t)$.

SOLUTION:

Our goal is to use Examples 1 – 4 to determine how the constants A, ω, h, and k affect the period, midline, amplitude, and horizontal shift of a sinusoidal function so let's start by observing what the values of A, ω, h, and k are in $g(t) = 2\sin(t) - 3$. It should be clear that function g is a sinusoidal function of the form $y = A\sin\left(\omega\left(t - h\right)\right) + k$ where $A = 2$, $\omega = 1$, $h = 0$, and $k = -3$.

After inspecting the rules for the functions f and g, we should notice that we could construct the function $g(t) = 2\sin(t) - 3$ by multiplying the outputs of the function $f(t) = \sin(t)$ by 2 and then subtracting 3 from the result. We can express this algebraically with the equation below:

$$g(t) = 2f(t) - 3$$

Based on what we know about graph transformations, we can conclude that we can obtain graph of g by starting with the graph of f and first stretching it vertically by a factor of 2 and then shifting it down 3 units. Since $f(t) = \sin(t)$ has amplitude 1 unit, if we stretch it vertically by a factor of 2 then we'll double the amplitude, so we should expect that the amplitude of g to be 2 units. Also, since $f(t) = \sin(t)$ has midline $y = 0$, when we shift it down 3 units to draw the graph of g, the resulting midline will be $y = -3$. (Note that since graphing g required no horizontal transformations of $f(t) = \sin(t)$, the graph of g must have the same period as the graph of $f(t) = \sin(t)$: 2π units.) Let's summarize what we've learned about $g(t) = 2\sin(t) - 3$:

period: 2π units

midline: $y = -3$

amplitude: 2 units

horizontal shift: 0 units

The graphs of $f(t) = \sin(t)$ and $g(t) = 2\sin(t) - 3$ are given in Figure 1 below.

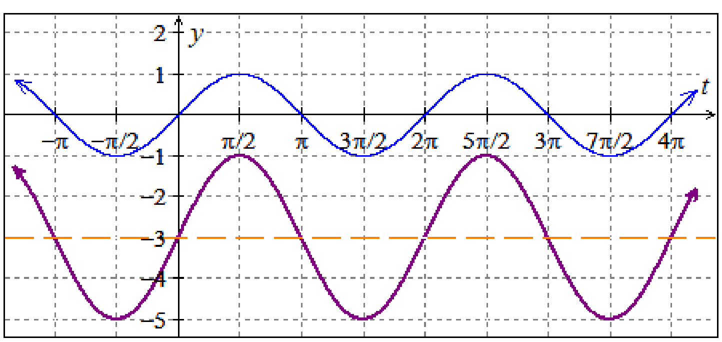


Figure 1: The graphs of $f(t) = \sin(t)$ and $g(t) = 2\sin(t) - 3$.

EXAMPLE 2: Describe how we can transform the graph of $f(t) = \sin(t)$ into the graph of $n(t) = \sin\left(t + \frac{\pi}{4}\right)$; state the period, midline, and amplitude of $y = n(t)$.

SOLUTION:

Notice that the function n is a sinusoidal function of the form $y = A\sin\left(\omega\left(t - h\right)\right) + k$ where $A = 1$, $\omega = 1$, $h = -\frac{\pi}{4}$, and $k = 0$.

After inspecting the rules for the functions f and n, it should be clear that we can write n in terms of f as follows: $n(t) = f\left(t + \frac{\pi}{4}\right)$. Based on what we know about graph transformations, we can conclude that we can obtain graph of n by starting with the graph of f and shifting it left $\frac{\pi}{4}$ units. Since a horizontal shift won't affect the period, midline, or amplitude, we should expect that the period, midline, and amplitude of $n(t) = \sin\left(t + \frac{\pi}{4}\right)$ are the same as $f(t) = \sin(t)$:

period: 2π units

midline: $y = 0$

amplitude: 1 unit

horizontal shift: $-\frac{\pi}{4}$ units

The graphs of $f(t) = \sin(t)$ and $n(t) = \sin\left(t + \frac{\pi}{4}\right)$ are given in Figure 2.

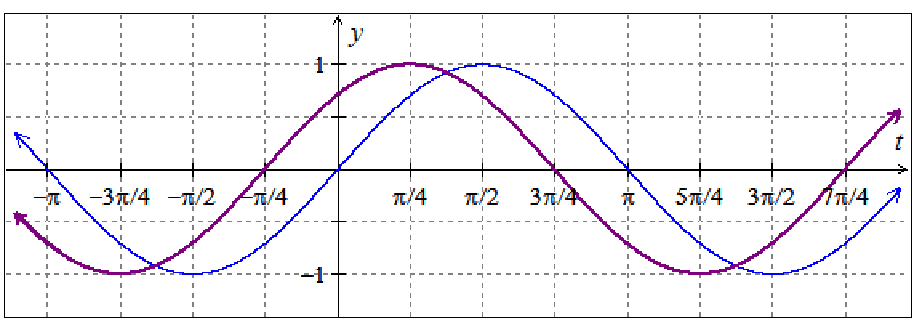


Figure 2: The graphs of $f(t) = \sin(t)$ (blue) and $n(t) = \sin\left(t + \frac{\pi}{4}\right)$ (purple).

EXAMPLE 3: Describe how we can transform the graph of $p(t) = \cos(t)$ into the graph of $q(t) = -\cos(2t)$ and find the period, midline, and amplitude of $y = q(t)$.

SOLUTION:

Notice that the function q is a sinusoidal function of the form $y = A\cos\big(\omega\big(t-h\big)\big)+k$ where $A=-1$, $\omega = 2$, $h = 0$, and $k = 0$.

After inspecting the rules for the functions p and q, it should be clear that we can write q in terms of p as follows: $q(t) = -p(2t)$. Based on what we know about graph transformations, we can conclude that we can obtain graph of q by starting with the graph of p and first stretching it horizontally by a factor of $\frac{1}{2}$ (i.e., compressing the graph by a factor of 2) and then reflecting it about the t-axis. Since $p(t) = \cos(t)$ has period 2π units, if we compress the graph by a factor of 2 then the period will be shrunk to π units. Since we aren't stretching the graph of p vertically, we should expect that the amplitude of q is the same as the amplitude of p: 1 unit. Also, since we aren't shifting the graph of p vertically, we should expect that the midline of q is the same as the midline of p: $y = 0$. Let's summarize what we've learned about $q(t) = -\cos(2t)$:

period: π units

midline: $y = 0$

amplitude: 1 unit

horizontal shift: 0 units

The graphs of $p(t) = \cos(t)$ and $q(t) = -\cos(2t)$ are given in Figure 3.

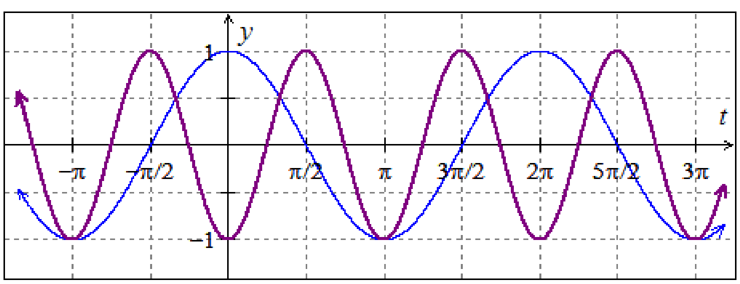


Figure 3: The graphs of $p(t) = \cos(t)$ (blue) and $q(t) = -\cos(2t)$ (purple).

Notice that the graph of $q(t) = -\cos(2t)$ completes two periods in the interval $[0, 2\pi]$. In general, the number ω in a sinusoidal function of the form $y = A\sin\big(\omega\big(t-h\big)\big)+k$ or $y = A\cos\big(\omega\big(t-h\big)\big)+k$ represents the number of periods (or "cycles") that the function completes on an interval of length 2π. This number ω is called the **angular frequency** of a sinusoidal function.

When we use sinusoidal functions to represent real-life situations, we often take the input variable to be a unit of *time*. Suppose that in the function $q(t) = -\cos(2t)$, t represents seconds. Since the input of the cosine function *must* be radians, the units of $\omega = 2$ must be "radians per second". This way,

$$2\,\frac{\text{radians}}{\cancel{\text{second}}}\cdot t\ \cancel{\text{seconds}} = 2t\ \text{radians},$$

which has the appropriate units for the input of the cosine function. So if t represents seconds, the **angular frequency** of $q(t) = -\cos(2t)$ is "2 radians per second".

Another way to obtain the unit of the angular frequency is to use what we noticed above: the number 2 in $q(t) = -\cos(2t)$ represents the number of cycles that the function completes on an interval of length 2π. Since a cycle is equivalent to a complete rotation around a circle, or 2π radians, two cycles is equivalent to 4π radians. If the input variable t represents seconds, then the angular frequency is

$$\frac{4\pi\ \text{radians}}{2\pi\ \text{seconds}} = 2\ \text{rad/sec.}$$

EXAMPLE 4: Describe how we can transform the graph of $p(t) = \cos(t)$ into the graph $m(t) = 3\cos\left(\frac{1}{2}\left(t - \frac{\pi}{3}\right)\right) + 5$. State the period, midline, and amplitude of $y = m(t)$.

SOLUTION:

Notice that the function w is a sinusoidal function of the form $y = A\cos\left(\omega\left(t - h\right)\right) + k$ where $A = 3$, $\omega = \frac{1}{2}$, $h = \frac{\pi}{3}$, and $k = 5$. After inspecting the rules for the functions p and w, it should be clear that we can write m in terms of p as follows: $m(t) = 3p\left(\frac{1}{2}\left(t - \frac{\pi}{3}\right)\right) + 5$. Based on what we know about graph transformations, we can conclude that we can obtain graph of m by starting with the graph of p and first stretching it horizontally by a factor of 2, then shifting it right $\frac{\pi}{3}$ units, then stretching it vertically by a factor 3, and finally shifting it up 5 units. Since $p(t) = \cos(t)$ has period 2π units, if we stretch the graph by a factor of 2 then the period will be stretched to 4π units. Similarly, if we stretch the graph of $p(t) = \cos(t)$ vertically by a factor of 3 then we'll triple the amplitude, so we should expect the amplitude of m to be 3 units. Also, since $p(t) = \cos(t)$ has midline $y = 0$, when we shift it up 5 units to draw the graph of m, the resulting midline will be $y = 5$. Since we are shifting the graph right $\frac{\pi}{3}$ units, the horizontal shift is $\frac{\pi}{3}$ units. Let's summarize what we've learned about $y = m(t)$:

period: 4π units

midline: $y = 5$

amplitude: 3 units

horizontal shift: $\frac{\pi}{3}$ units

The graphs of $p(t) = \cos(t)$ and $m(t) = 3\cos\left(\frac{1}{2}\left(t - \frac{\pi}{3}\right)\right) + 5$ are given in Figure 4.

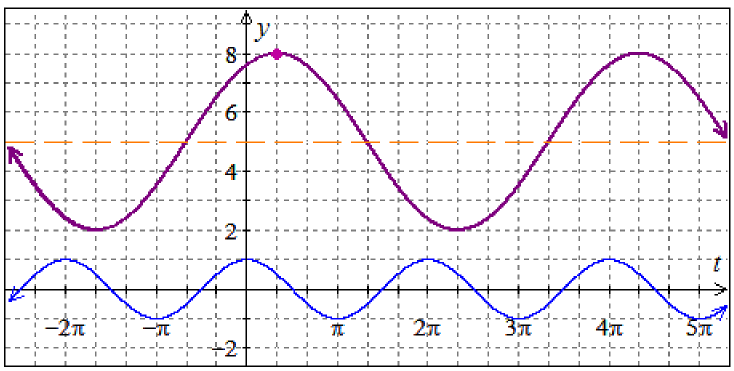


Figure 4: The graphs of $p(t) = \cos(t)$ (blue) and $m(t) = 3\cos\left(\frac{1}{2}\left(t - \frac{\pi}{3}\right)\right) + 5$ (purple).

Notice that the graph of $m(t) = 3\cos\left(\frac{1}{2}\left(t - \frac{\pi}{3}\right)\right) + 5$ completes one-half of a period (or "cycle") in the interval $[0, 2\pi]$. If we let the input variable, t, represent seconds, then $m(t) = 3\cos\left(\frac{1}{2}\left(t - \frac{\pi}{3}\right)\right) + 5$ completes one-half of a cycle every 2π seconds. Since one-half of a cycle is equivalent to half of a rotation around a circle, or π radians, then the **angular frequency** of the function m is

$$\frac{\pi \text{ radians}}{2\pi \text{ seconds}} = \frac{1}{2} \text{ rad/sec.}$$

Based on what we learned in the examples above, we can summarize the affect of the constants A, ω, h, and k on the period, midline, amplitude, and horizontal shift of functions of the form $y = A\sin\big(\omega(t-h)\big)+k$ and $y = A\cos\big(\omega(t-h)\big)+k$.

SUMMARY: Graphs of Sinusoidal Functions

The graphs of the sinusoidal functions

$$y = A\sin\big(\omega(t-h)\big)+k \quad \text{and} \quad y = A\cos\big(\omega(t-h)\big)+k$$

(where $A, \omega, h, k \in \mathbb{R}$) have the following properties:

period: $\frac{2\pi}{|\omega|}$ units

midline: $y = k$

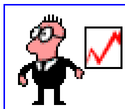
amplitude: $|A|$ units

horizontal shift: h units

angular frequency: ω radians per unit of t

EXAMPLE 5: Sketch a graph of $f(t) = 2\sin\left(\pi t - \frac{\pi}{4}\right) - 3$.

SOLUTION:



CLICK HERE to see a video of this example.

In order to use what we've just studied about functions of the form $y = A\sin\big(\omega(t-h)\big)+k$, we need to write the given function in this form, i.e., we need to factor π (which is playing the role of "ω") out of the input expression "$\pi t - \frac{\pi}{4}$":

$$\begin{aligned} f(t) &= 2\sin\left(\pi t - \tfrac{\pi}{4}\right) - 3 \\ &= 2\sin\left(\pi\left(t - \tfrac{1}{4}\right)\right) - 3 \end{aligned}$$

It should be clear that $f(t) = 2\sin\left(\pi\left(t - \frac{1}{4}\right)\right) - 3$ is a sinusoidal function of the form $y = A\sin\left(\omega(t-h)\right) + k$ where $A = 2$, $\omega = \pi$, $h = \frac{1}{4}$, and $k = -3$. Using what we found above, we can find the period, midline, amplitude, and horizontal shift of $y = f(t)$:

period: $\frac{2\pi}{|\omega|} = \frac{2\pi}{|\pi|} = 2$ units

midline: $y = -3$

amplitude: $|2| = 2$ units

horizontal shift: $\frac{1}{4}$ of a unit

We can use this information to sketch a graph of $f(t) = 2\sin\left(\pi\left(t - \frac{1}{4}\right)\right) - 3$; see Figure 5 below. (Note that the horizontal shift tells us where to "start" our usual sine wave.)

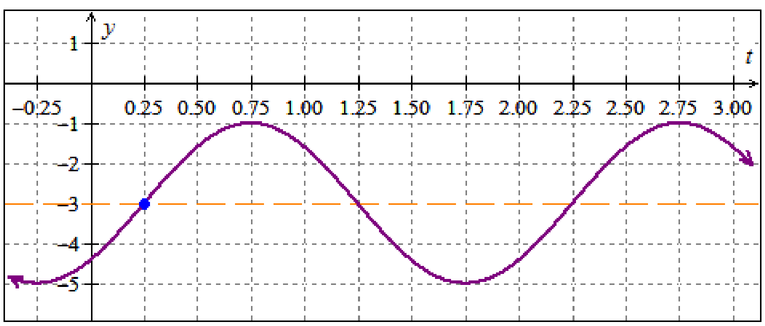


Figure 5: The graph of $f(t) = 2\sin\left(\pi\left(t - \frac{1}{4}\right)\right) - 3$. The blue point represents where we "start" our sine wave since the horizontal shift is $\frac{1}{4}$ of a unit.

Note that, according to what we discussed in Examples 3 and 4, if we let t represent seconds then we could state that the **angular frequency** of $f(t) = 2\sin\left(\pi\left(t - \frac{1}{4}\right)\right) - 3$ is π radians per second. Since π radians represents one-half of a rotation around a circle, the angular frequency "π radians per second t," is equivalent to one-half of a cycle per second. Notice that our graph in Figure 5 shows a function that completes one-half of a period in one unit of t!

EXAMPLE 6: Find two different algebraic rules (one involving sine and one involving cosine) for the function $y = g(t)$ graphed in Figure 6.

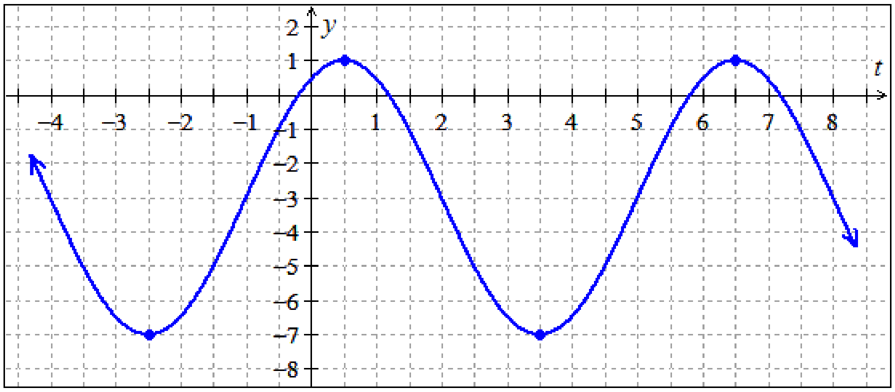
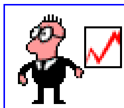


Figure 6: The graph of $y = g(t)$.

SOLUTION:



CLICK HERE to see a video of this example.

First let's write a rule involving sine, so our rule will have the form $g(t) = A\sin\left(\omega\left(t - h\right)\right) + k$ and we need to determine the values of A, ω, h, and k.

- The midline is the line midway between the function's maximum and minimum output values. The function's maximum output value is 1 and its minimum output value is -7. Since -3 is the average of these values, the midline is $y = -3$ so $k = -3$.

- The amplitude is the distance between the function's maximum output value, 1, and its midline $y = -3$, which is 4 units. Therefore, $|A| = 4$.

- The function completes one period between $t = -1$ and $t = 5$. Thus, the period of the function is $5 - (-1) = 6$. To find ω we need to solve $6 = 2\pi \cdot \frac{1}{\omega}$:

$$6 = 2\pi \cdot \frac{1}{\omega}$$
$$\Rightarrow \quad \omega = 2\pi \cdot \frac{1}{6}$$
$$\Rightarrow \quad \omega = \frac{\pi}{3}$$

- We know that, near y-axis, the graph of $y = \sin(t)$ is increasing and passes through its midline: since we want to use sine as our 'root' function, we need to look for a spot in the graph of $y = g(t)$ where it shows this behavior. One such spot is at $t = -1$ so we can view the graph of $y = g(t)$ as a sine wave shifted left 1 unit and use $h = -1$.

Therefore, an algebraic rule for g is $g(t) = 4\sin\left(\frac{\pi}{3}\left(t - (-1)\right)\right) - 3$, which we can simplify as $g(t) = 4\sin\left(\frac{\pi}{3}\left(t + 1\right)\right) - 3$. (Note that $g(t) = 4\sin\left(\frac{\pi}{3}\left(t - 5\right)\right) - 3$ is another possibility.)

Now we'll write a rule involving cosine, so our rule will have the form $g(t) = A\sin\left(\omega\left(t - h\right)\right) + k$. Since the amplitude, period, and midline aren't dependent on whether we use sine or cosine in our algebraic rule, we can use the same values for A, ω, and k that we used above. So we only need to determine an appropriate horizontal shift, h, that works for cosine. We know that the graph of $y = \cos(t)$ reaches its maximum value at the y-axis: since we want to use cosine as our 'root' function, we need to look for a spot in the graph of $y = g(t)$ where it reaches its maximum. One such spot is at $t = \frac{1}{2}$ so we can view the graph of $y = g(t)$ as a cosine wave shifted right $\frac{1}{2}$ of a unit and use $h = \frac{1}{2}$. Therefore, an algebraic rule g is $g(t) = 4\cos\left(\frac{\pi}{3}\left(t - \frac{1}{2}\right)\right) - 3$. (Note that $g(t) = 4\cos\left(\frac{\pi}{3}\left(t - 6.5\right)\right) - 3$ is another possibility.)