# Section I: The Trigonometric Functions

## Chapter 3, Part 3: Intro to the Trigonometric Functions

Now let's determine the sine and cosine of some important angles, namely, $30^\circ$, $45^\circ$, and $60^\circ$ (i.e., $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$). We focus on these angles since we can use some basic geometry to easily find their sine and cosine values – but we cannot easily find the sine and cosine of most other angles. During this course, we'll refer to these angles (and multiples of these angels) as *friendly angles* since their sine and cosine values are easy to find. You ***need*** to learn (or "memorize") these values.

Let's start with $30^\circ$ (or $\frac{\pi}{6}$). Since we want to find $\sin(30^\circ)$ and $\cos(30^\circ)$, we need to find the horizontal and vertical coordinates of the point $P$ on the circumference of the unit circle specified by the angle $30^\circ$; see Figure 1.

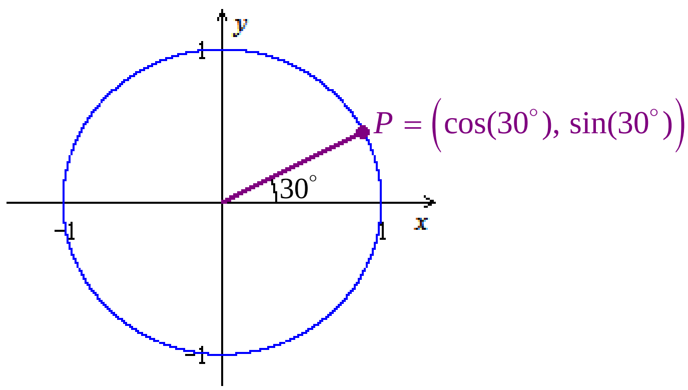


**Figure 1**

Notice that we can use the $30^\circ$ angle and the terminal side of the $30^\circ$ angle (i.e., the radius of the unit circle) in Figure 1 to construct a right-triangle with a $1$ unit long hypotenuse and side-lengths that are the horizontal and vertical coordinates of the point $P$; see Figure 2. If we can find the side-lengths of the triangle in Figure 2, we will have found $\sin(30^\circ)$ and $\cos(30^\circ)$.

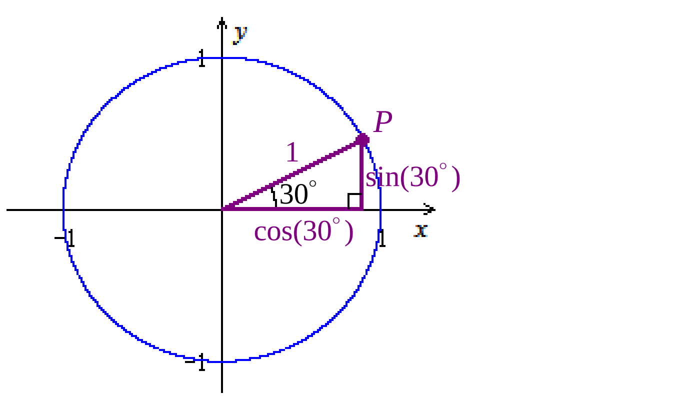


**Figure 2**

In Figure 3, we've magnified the triangle from Figure 2. Notice that, since the sum of the angles in a triangle is always $180^\circ$ and since this triangle already has a $30^\circ$ angle and a $90^\circ$ angle, the third angle must be $60^\circ$.

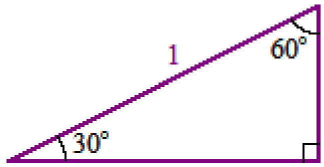


**Figure 3**

In order to find the lengths of the sides of this triangle, in Figure 4 we've placed a "mirror image" of the triangle under the triangle to form a larger triangle.

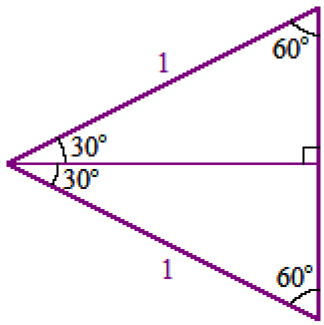


**Figure 4**

Notice that this larger triangle is equiangular since all three of its angles have the same measure: $60^\circ$; see Figure 5. As you may have studied previously in a geometry course, equiangular triangles are also equilateral, i.e., they contain three equal sides. Since two of the sides are each $1$ unit long, the other side must also be $1$ unit long.

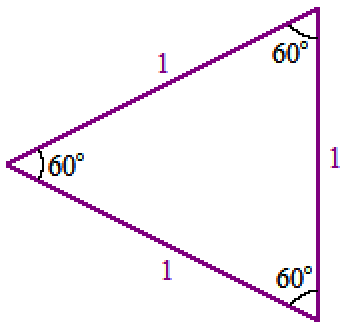


**Figure 5**

Since the sides of the triangle are each $1$ unit long, if we cut one of the sides in half, each part will be $\frac{1}{2}$ of a unit. Notice how, in Figure 4, the horizontal segment that creates the two $30^\circ$ angles cuts the side opposite these angles in half; see Figure 6.

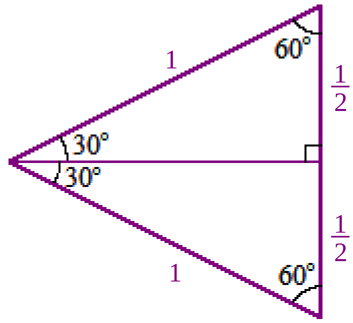


**Figure 6**

Now we return to the original right triangle that we were looking at in Figure 3, and label in Figure 7 all of the information we've found so far. We've labeled the unknown side $a$.

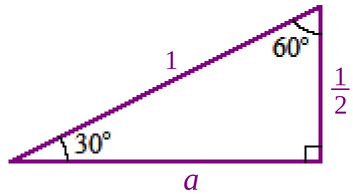


**Figure 7**

We can now use the *Pythagorean Theorem* (see the **green box** below) to find $a$.

$$
\begin{aligned}
& a^2 + \left(\tfrac{1}{2}\right)^2 = 1^2 \\
\Rightarrow \quad & a^2 + \tfrac{1}{4} = 1 \\
\Rightarrow \quad & a^2 = \tfrac{3}{4} \\
\Rightarrow \quad & a = \tfrac{\sqrt{3}}{2}
\end{aligned}
$$

In Figure 8, we've labeled all of the side-lengths of this triangle:

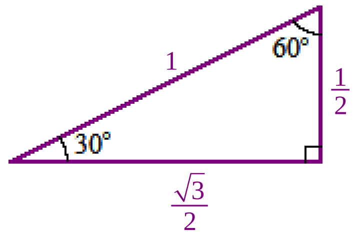


**Figure 8**

**THE PYTHAGOREAN THEOREM:**

If the sides of a right triangle (i.e., a triangle with a $90^\circ$ angle) are labeled like the one given in Figure 9, then $\boldsymbol{a}^2 + \boldsymbol{b}^2 = \boldsymbol{c}^2$.

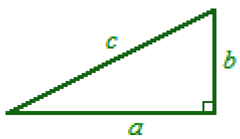


**Figure 9**

Now we can put this triangle back inside the unit circle; see Figure 10.

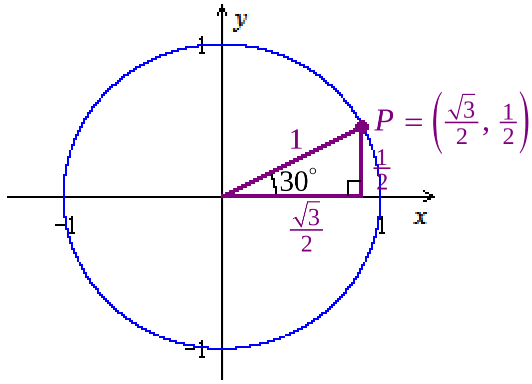


**Figure 10**

Recall that the coordinates of $P$ are $\left(\cos(30^\circ), \sin(30^\circ)\right)$. Thus,

$$\mathbf{\sin(30^\circ) = \frac{1}{2}} \quad \text{and} \quad \mathbf{\cos(30^\circ) = \frac{\sqrt{3}}{2}}$$

Now let's find the sine and cosine values for $60^\circ$ (or $\frac{\pi}{3}$) which are the coordinates of the point $Q$ on the circumference of the unit circle specified by the angle $60^\circ$; see Figure 11.

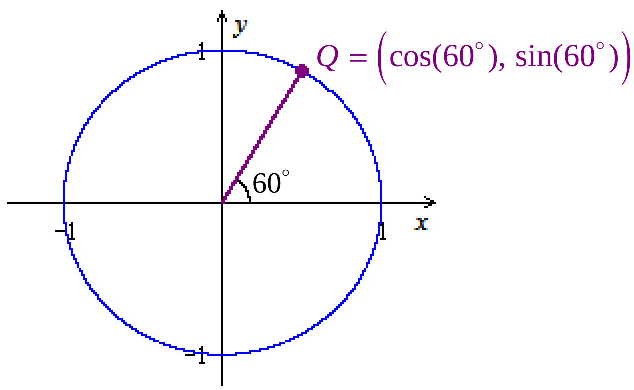


**Figure 11**

As we did with $30^\circ$, we can use the $60^\circ$ angle in Figure 11 to construct a right-triangle that we can use to find the coordinates of the point $Q$; see Figure 12.

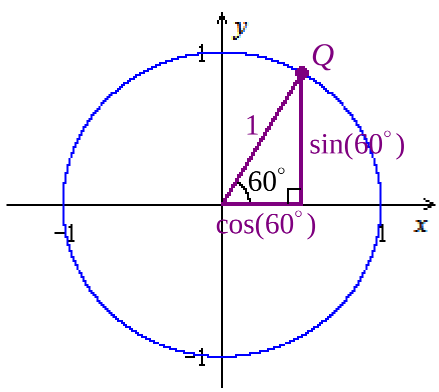


**Figure 12**

In Figure 13, we've magnified the triangle from Figure 12. Notice that, since the sum of the angles in a triangle is always $180^\circ$ and since this triangle already has a $60^\circ$ angle and a $90^\circ$ angle, the third angle must be $30^\circ$.

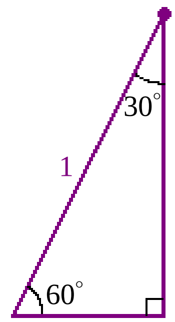


**Figure 13**

Notice that this is essentially the same triangle that we studied in Figure 8, just with a different orientation. So we can use the triangle in Figure 8 to obtain the lengths of the sides of this triangle; see Figure 14.

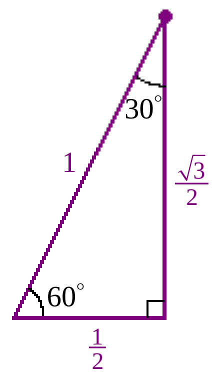


**Figure 14**

Now we can put this triangle back inside the unit circle; see Figure 15.

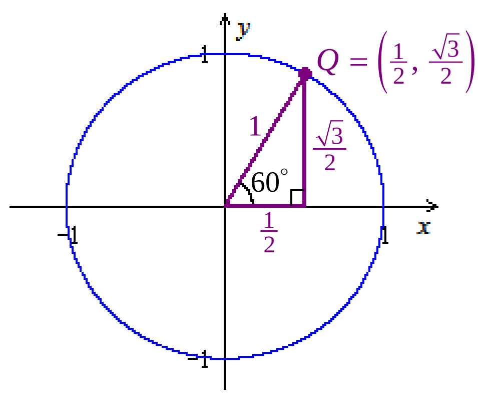


**Figure 15**

Recall that the coordinates of $P$ are $\left(\cos(60^\circ), \sin(60^\circ)\right)$. Therefore,

$$\mathbf{\sin(60^\circ) = \frac{\sqrt{3}}{2}} \quad \textbf{and} \quad \mathbf{\cos(60^\circ) = \frac{1}{2}}$$

Now let's find the sine and cosine values for $45^\circ$ (or $\frac{\pi}{4}$) which are the coordinates of the point $M$ on the circumference of the unit circle specified by the angle $45^\circ$; see Figure 16.

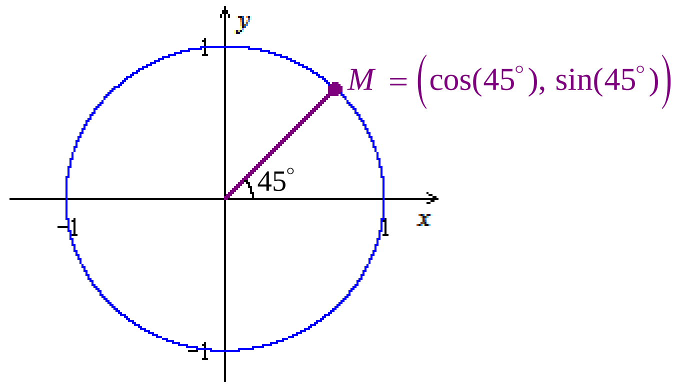


**Figure 16**

As we did with $30^\circ$ and $60^\circ$, we can use the $45^\circ$ angle in Figure 16 to construct a right-triangle that we can use to find the coordinates of the point $M$; see Figure 17.

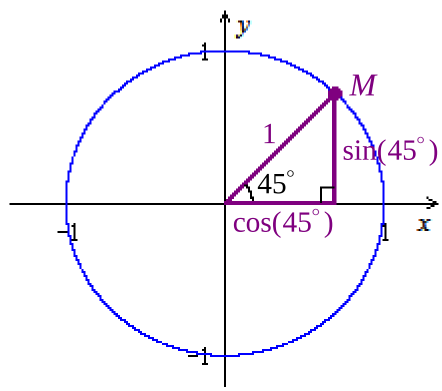


**Figure 17**

In Figure 18, we've magnified the triangle from Figure 17. Notice that, since the sum of the angles in a triangle is always $180^\circ$ and since this triangle already has a $45^\circ$ angle and a $90^\circ$ angle, the third angle must also be $45^\circ$.

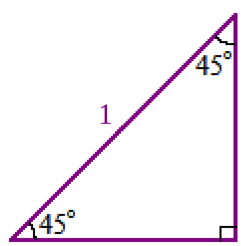


**Figure 18**

This is an *isosceles* triangle since two of the angles are equal; the sides opposite the equal angles must also be of equal length. (This is a property of isosceles triangles.) In Figure 19, we've used the same symbol, $a$, to label the lengths of these two sides.

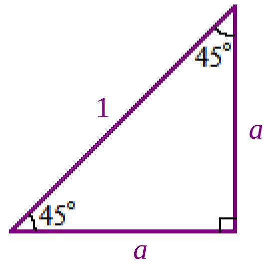


**Figure 19**

We can now use the Pythagorean Theorem to find $a$.

$$a^2 + a^2 = 1^2$$
$$\Rightarrow \quad 2a^2 = 1$$
$$\Rightarrow \quad a^2 = \frac{1}{2}$$
$$\Rightarrow \quad a = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

In Figure 20, we've labeled the side-lengths of this triangle:

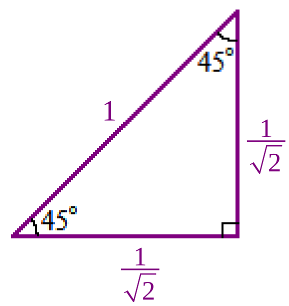


**Figure 20**

Now we can put this triangle back inside the unit circle; see Figure 21.

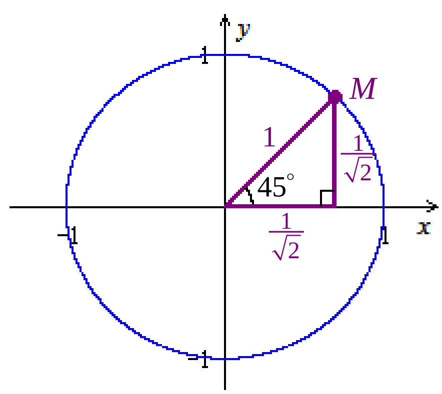


**Figure 21**

Recall that the coordinates of $M$ are $\left(\cos(45^\circ), \sin(45^\circ)\right)$. Therefore,

$$\mathbf{sin(45^\circ) = \frac{1}{\sqrt{2}}} \quad \textbf{and} \quad \mathbf{cos(45^\circ) = \frac{1}{\sqrt{2}}}$$

Note that although there is nothing wrong with the expression $\frac{1}{\sqrt{2}}$, in the "old days" (i.e., the pre-calculator era), people didn't like to have irrational numbers (i.e., numbers like $\sqrt{2}$) in the denominator of fractions since the tables they used to help them approximate values didn't contain approximations for expressions with irrational numbers in the denominator. As a result, there is a procedure known as "rationalizing the denominator" which allows us to get rid of the radicals in the denominator. (You should have studied this in an Intermediate Algebra course.) When you study trigonometry, you often see the rationalized form of the number so it's worth taking note of the rationalized form of $\frac{1}{\sqrt{2}}$:

$$\begin{aligned}\frac{1}{\sqrt{2}} &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

Let's put the values for sine and cosine that we've found so far in a table. **It is important that you learn these values**. In Part 4 of Chapter 3, we'll discuss a strategy for remembering these values.

| $\theta$ (degrees) | $30^\circ$ | $45^\circ$ | $60^\circ$ |
|---|---|---|---|
| $\theta$ (radians) | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ |
| $\cos(\theta)$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ |
| $\sin(\theta)$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ |

Recall from Part 2 of Chapter 3 the following definitions of the "other trig functions." In the next three examples we'll find the of tangent, secant, cosecant, and cotangent of $30^\circ$, $45^\circ$, and $60^\circ$.

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\cot(t) = \frac{1}{\tan(\theta)}$$

**EXAMPLE 1:** Find $\tan\left(\frac{\pi}{6}\right)$, $\sec\left(\frac{\pi}{6}\right)$, $\csc\left(\frac{\pi}{6}\right)$, and $\cot\left(\frac{\pi}{6}\right)$.

SOLUTION:

$$\tan\left(\frac{\pi}{6}\right) = \frac{\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)} \quad \text{(using the definition of tangent)}$$

$$= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} \quad \text{(since } \sin\left(\tfrac{\pi}{6}\right) = \tfrac{1}{2} \text{ and } \cos\left(\tfrac{\pi}{6}\right) = \tfrac{\sqrt{3}}{2}\text{)}$$

$$= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \quad \text{(either form of this number is acceptable)}$$

$$\sec\left(\frac{\pi}{6}\right) = \frac{1}{\cos\left(\frac{\pi}{6}\right)} \quad \text{(using the definition of secant)}$$

$$= \frac{1}{\frac{\sqrt{3}}{2}} \quad \text{(since } \cos\left(\tfrac{\pi}{6}\right) = \tfrac{\sqrt{3}}{2}\text{)}$$

$$= \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \quad \text{(either form of this number is acceptable)}$$

$$\csc\left(\frac{\pi}{6}\right) = \frac{1}{\sin\left(\frac{\pi}{6}\right)} \quad \text{(using the definition of cosecant)}$$

$$= \frac{1}{\frac{1}{2}} \quad \text{(since } \sin\left(\tfrac{\pi}{6}\right) = \tfrac{1}{2}\text{)}$$

$$= 2$$

$$\cot\left(\frac{\pi}{6}\right) = \frac{\cos\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{6}\right)} \quad \text{(using the definition of cotangent)}$$

$$= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \quad \text{(since } \sin\left(\tfrac{\pi}{6}\right) = \tfrac{1}{2} \text{ and } \cos\left(\tfrac{\pi}{6}\right) = \tfrac{\sqrt{3}}{2}\text{)}$$

$$= \sqrt{3}$$

**EXAMPLE 2:** Find $\tan\left(\frac{\pi}{4}\right)$, $\sec\left(\frac{\pi}{4}\right)$, $\csc\left(\frac{\pi}{4}\right)$, and $\cot\left(\frac{\pi}{4}\right)$.

SOLUTION:

$$\tan\left(\frac{\pi}{4}\right) = \frac{\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)}$$ (using the definition of tangent)

$$= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$$ (since $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$)

$$= 1$$

$$\sec\left(\frac{\pi}{4}\right) = \frac{1}{\cos\left(\frac{\pi}{4}\right)}$$ (using the definition of secant)

$$= \frac{1}{\frac{\sqrt{2}}{2}}$$ (since $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$)

$$= \frac{2}{\sqrt{2}} = \sqrt{2}$$ (either form of this number is acceptable)

$$\csc\left(\frac{\pi}{4}\right) = \frac{1}{\sin\left(\frac{\pi}{4}\right)}$$ (using the definition of cosecant)

$$= \frac{1}{\frac{\sqrt{2}}{2}}$$ (since $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$)

$$= \frac{2}{\sqrt{2}} = \sqrt{2}$$ (either form of this number is acceptable)

$$\cot\left(\frac{\pi}{4}\right) = \frac{\cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)}$$ (using the definition of cotangent)

$$= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$$ (since $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$)

$$= 1$$

**EXAMPLE 3:** Find $\tan\left(\frac{\pi}{3}\right)$, $\sec\left(\frac{\pi}{3}\right)$, $\csc\left(\frac{\pi}{3}\right)$, and $\cot\left(\frac{\pi}{3}\right)$.

SOLUTION:

$$\tan\left(\frac{\pi}{3}\right) = \frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)}$$ (using the definition of tangent)

$$= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$ (since $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$)

$$= \sqrt{3}$$

$$\sec\left(\frac{\pi}{3}\right) = \frac{1}{\cos\left(\frac{\pi}{3}\right)}$$ (using the definition of secant)

$$= \frac{1}{\frac{1}{2}}$$ (since $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$)

$$= 2$$

$$\csc\left(\frac{\pi}{3}\right) = \frac{1}{\sin\left(\frac{\pi}{3}\right)}$$ (using the definition of cosecant)

$$= \frac{1}{\frac{\sqrt{3}}{2}}$$ (since $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$)

$$= \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$ (either form of this number is acceptable)

$$\cot\left(\frac{\pi}{3}\right) = \frac{\cos\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\pi}{3}\right)}$$ (using the definition of cotangent)

$$= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$ (since $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$)

$$= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$ (either form of this number is acceptable)