Section I: The Trigonometric Functions

Chapter 3, Part 1: Intro to the Trigonometric Functions

In Example 4 in Section I: Chapter 2, we observed that a circle rotating about its center (i.e., a Ferris wheel) lends itself naturally to the study of periodic functions. In fact, the two most important *trigonometric functions* are defined in terms of a unit circle: the **sine** and **cosine** functions.

DEFINITION: The **sine function**, denoted $\mathbf{sin}(\boldsymbol{\theta})$, associates each angle θ with the vertical coordinate (i.e., the y-coordinate) of the point P specified by the angle θ on the circumference of a unit circle.

The **cosine function**, denoted $\mathbf{cos}(\boldsymbol{\theta})$, associates each angle θ with the horizontal coordinate (i.e., the x-coordinate) of the point P specified by the angle θ on the circumference of a unit circle.

So the point P in Figure 1 has coordinates $(x,\ y) = (\cos(\theta),\ \sin(\theta))$.

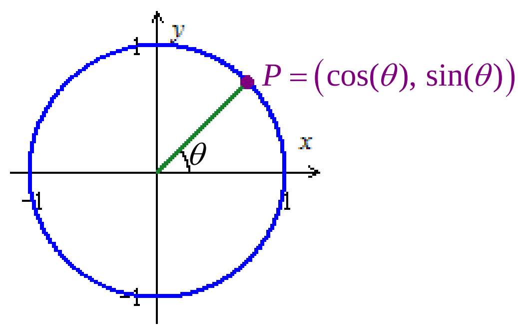

Figure 1

There are four other trigonometric functions. These four functions are defined in terms of the sine and cosine functions so first let's get familiar with sine and cosine. Later in this chapter we'll define the four other trigonometric functions.

EXAMPLE 1: The angle θ specifies the point $P = \left(-\frac{3}{5},\ \frac{4}{5}\right)$ on the circumference of a unit circle; see Figure 2. Find $\sin(\theta)$ and $\cos(\theta)$.

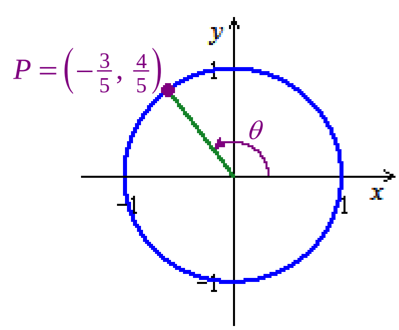

Figure 2

SOLUTION:

$$\sin(\theta) = \tfrac{4}{5} \quad \text{and} \quad \cos(\theta) = -\tfrac{3}{5}$$

In order to enhance our understanding of the sine and cosine functions, we should determine some particular values for the functions and sketch their graphs. But before we confront these details, let's determine the signs (positive or negative) of the sine and cosine functions in the four different quadrants of the coordinate plane. (To review the quadrants of the coordinate plane, see Section I: Chapter 1, Figure 6.)

- When the terminal side of angle θ is in Quadrant I, both the x- and y-coordinates of point P are positive, so

  θ is in Quadrant I means that $\cos(\theta) > 0$ and $\sin(\theta) > 0$.

- When the terminal side of angle θ is in Quadrant II, the y-coordinate of point P is positive but the x-coordinate is negative, so

  θ is in Quadrant II means that $\cos(\theta) < 0$ and $\sin(\theta) > 0$.

- When the terminal side of angle θ is in Quadrant III, both the x- and y-coordinates of point P are negative, so

  θ is in Quadrant III means that $\cos(\theta) < 0$ and $\sin(\theta) < 0$.

- When the terminal side of angle θ is in Quadrant IV, the x-coordinate of point P is positive but the y-coordinate is negative, so

  θ is in Quadrant IV means that $\cos(\theta) > 0$ and $\sin(\theta) < 0$.

Let's summarize in Figure 3 what we've determined about the signs of the sine and cosine functions in the different quadrants:

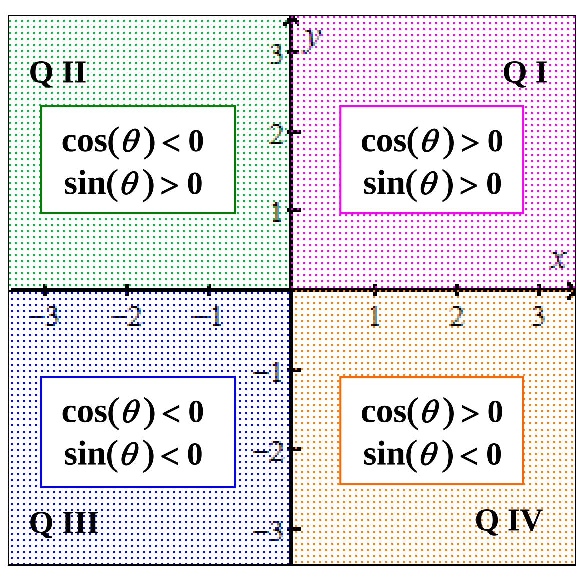

Figure 3: The signs of sine and cosine in the four quadrants.

Now let's find the sine and cosine of a few particular angles. Recall that the sine and cosine functions represent the coordinates of points on a unit circle, and the easiest points for us to find on the unit circle are points where the circumference of the circle intersects the coordinate axes; let's start by finding the corresponding sine and cosine values. Keep in mind that **cosine** represents the ***x*-coordinate** and **sine** represents the ***y*-coordinate**.

- The angle $\theta = 90^\circ$, i.e., $\theta = \frac{\pi}{2}$ radians, specifies the point $(0, 1)$ on the circumference of a unit circle; see Figure 4a.

  Thus, $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$.

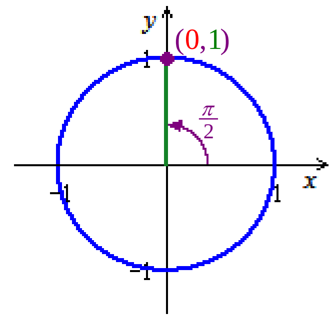

Figure 4a

- The angle $\theta = 180^\circ$, i.e., $\theta = \pi$ radians, specifies the point $(-1, 0)$ on the circumference of a unit circle; see Figure 4b.

  Thus, $\cos(\pi) = -1$ and $\sin(\pi) = 0$.

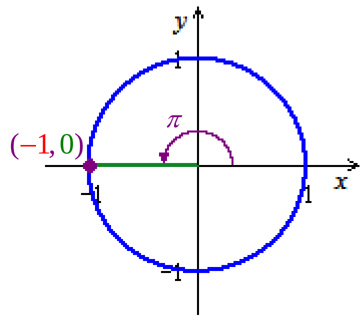

Figure 4b

- The angle $\theta = 270^\circ$, i.e. $\theta = \frac{3\pi}{2}$ radians, specifies the point $(0, -1)$ on the circumference of a unit circle; see Figure 4c.

  Thus, $\cos\left(\frac{3\pi}{2}\right) = 0$ and $\sin\left(\frac{3\pi}{2}\right) = -1$.

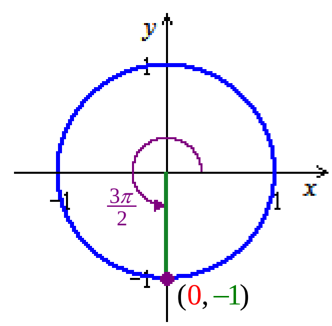

Figure 4c

- The angle $\theta = 360^\circ$, i.e., $\theta = 2\pi$ radians, specifies the point $(1, 0)$ on the circumference of a unit circle; see Figure 4d.

  Thus, $\cos(2\pi) = 1$ and $\sin(2\pi) = 0$.

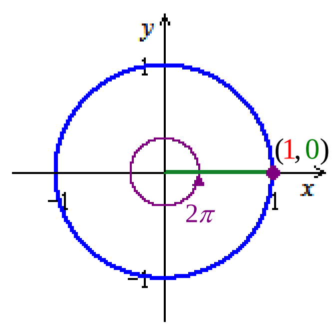

Figure 4d

Notice that angles of measure 2π radians (i.e., $\theta = 360^\circ$) and 0 radians specify the same point: $(1, 0)$. Thus, the sine and cosine values for 2π radians and 0 radians are the same, i.e.,

$$\cos(2\pi) = \cos(0) = 1 \quad \text{and} \quad \sin(2\pi) = \sin(0) = 0.$$

Since ANY angle θ and $\theta + 2\pi$ specify the *same* point on the unit circle, the sine and cosine values of θ and $\theta + 2\pi$ are the same; therefore, the period of the sine and cosine functions is 2π radians.

For all θ, $\sin(\theta) = \sin(\theta + 2\pi)$ and $\cos(\theta) = \cos(\theta + 2\pi)$ so the **period** of both $s(\theta) = \sin(\theta)$ and $c(\theta) = \cos(\theta)$ is 2π radians (i.e., 360°).

Now we'll sketch graphs of the sine and cosine functions. Let's start by organizing the function values we determined above in a table:

θ (degrees)	0°	90°	180°	270°	360°	450°	540°	630°	720°
θ (radians)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
$y = \cos(\theta)$	1	0	-1	0	1	0	-1	0	1
$y = \sin(\theta)$	0	1	0	-1	0	1	0	-1	0

In Figure 5a and 5b we've plotted the information in the table on two coordinate planes.

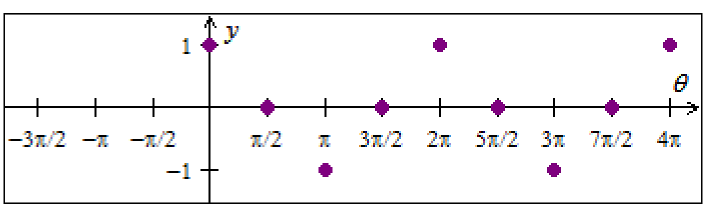

Figure 5a: Some points on $y = \cos(\theta)$.

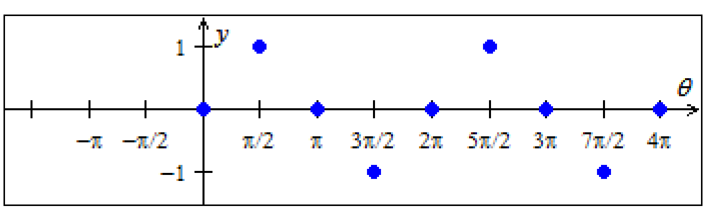

Figure 5b: Some points on $y = \sin(\theta)$.

In the next chapters we'll find many more values of sine and cosine, but we can go ahead and connect the dots in these graphs to obtain reasonable sketches of the graphs of the sine and cosine functions in Figures 6a and 6b. (We can use what we observed in Example 4 from Section I: Chapter 2 when we studied the Ferris wheel.)

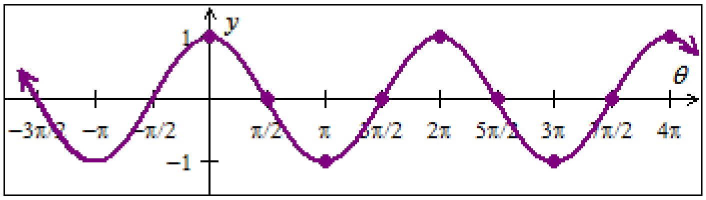

Figure 6a: The graph of $y = \cos(\theta)$.

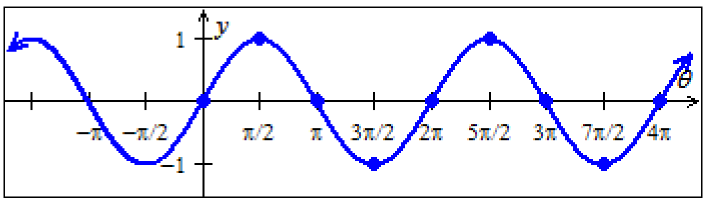

Figure 6b: The graph of $y = \sin(\theta)$.

KEY POINT: To get these graphs on your calculator, be sure to change the **angle mode** of to **radians**. If you want to get graphs of sine and cosine in **degree mode**, be sure the **window** has a horizontal interval like $[-300, 1440]$ since the period of the function is $360°$.

Notice that the graphs of $y = \cos(\theta)$ and $y = \sin(\theta)$ are very similar. In fact, if we shift $y = \sin(\theta)$ to the left $\frac{\pi}{2}$ units, we'll obtain the graph of $y = \cos(\theta)$. Using what we learned about graph transformation in MTH 111, this means that $\mathbf{\cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)}$. Similarly, if we shift $y = \cos(\theta)$ to the right $\frac{\pi}{2}$ units, we'll obtain the graph of $y = \sin(\theta)$. Using what we know about graph transformation, this means that $\mathbf{\sin(\theta) = \cos\left(\theta - \frac{\pi}{2}\right)}$.

The two bold equations above are called **identities** since the left and right sides of the equations are *always* identical, no matter what value of θ is used.

DEFINITION: An **identity** is an equation that is true for all values in the domains of the involved expressions.

Earlier in this chapter we observed a couple of identities but didn't call them identities. The equations

$$\sin(\theta) = \sin(\theta + 2\pi) \quad \text{and} \quad \cos(\theta) = \cos(\theta + 2\pi)$$

are identities since they are true for *all* values of θ. We can use the definitions and graphs of sine and cosine to determine a few other important identities. Do your best to convince yourself that each of following identities is true. I strongly encourage you to graph (on your graphing calculator) both sides of the identities and notice that the graphs are identical. Also, use what you learned in MTH 111 about graph transformations and symmetry to make sense of WHY these identities are true. (In Section I: Chapter 6 we will review graph transformations.)

SOME IDENTITIES

- $\sin(\theta) = \sin(\theta + 2\pi)$
- $\cos(\theta) = \cos(\theta + 2\pi)$
- $\cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)$
- $\sin(\theta) = \cos\left(\theta - \frac{\pi}{2}\right)$
- $\cos(-\theta) = \cos(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\sin(\theta) = \sin(\pi - \theta)$

The last identity on this list, $\sin(\theta) = \sin(\pi - \theta)$, is hardest one to make sense of, but it is worth taking time to understand it since it is a useful identity. (We will use it in Section I: Chapter 9 when we solve equations involving the sine function.) The easiest way to understand it is to focus on the θ-values between 0 and $\frac{\pi}{2}$. If θ is in this interval, then it should be clear (if you study the graph of $y = \sin(\theta)$) that $\sin(\pi - \theta)$ is the same as $\sin(\theta)$ due to the symmetry of the sine function between 0 and π. Once you see why $\sin(\theta) = \sin(\pi - \theta)$ for θ-values between 0 and $\frac{\pi}{2}$, it will be easier to convince yourself that the identity holds for all values of θ.

We'll study a few more important identities at the end of this chapter and we'll study *proving* trigonometric identities in Section II: Chapter 1.

Although the sine and cosine functions are defined via the *unit* circle, we can use sine and cosine to find the coordinates of a point on the circumference of any circle. First, let's notice a few things about the unit circle.

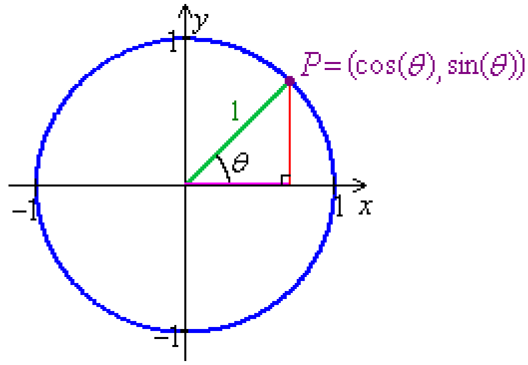

Figure 7: The unit circle with a point P specified by the angle θ.

As shown in Figure 7, we can construct a right-triangle using the terminal side of angle θ and the horizontal and vertical components of the point P. By construction, this right triangle has a hypotenuse of length 1 unit (since this is the radius), a horizontal component of length $\cos(\theta)$, and a vertical component of length $\sin(\theta)$; see Figure 8. Now let's consider a circle with a different radius; see Figure 9. Keep in mind that the angle θ is the same in Figure 9 as it was in Figures 7 and 8.

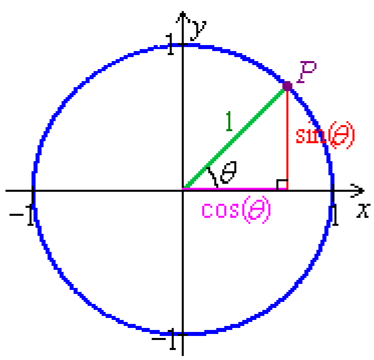

Figure 8: The right triangle induced by point P on the unit circle.

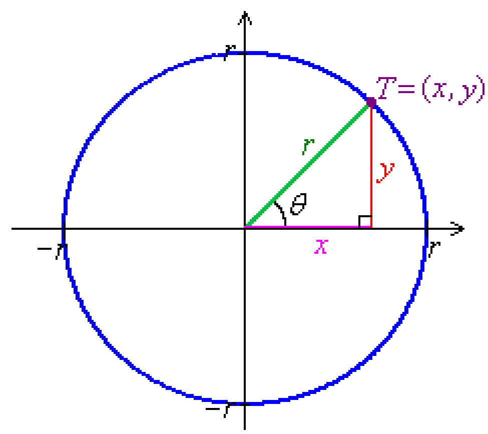

Figure 9: The right triangle induced by point T on a circle of radius r.

Since the two triangles in Figures 8 and 9 are both right triangles (i.e., they both have a 90° angle) and both have an angle θ, basic properties of geometry can be used to prove that the two triangles are **similar**; see Figure 10.

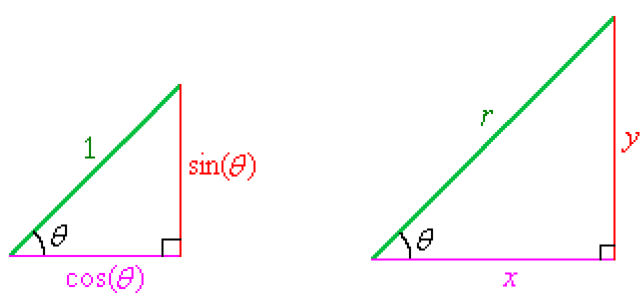

Figure 10: Similar right triangles with angle θ.

A well-known fact about similar triangles is that the ratio of side-lengths is constant. For example, the ratio of the height to the hypotenuse of the respective triangles is constant. Similarly, the ratio of the horizontal length to the hypotenuse of the respective triangles is constant. We can use these facts and the triangles in Figure 10 to obtain the following equations:

$$\frac{\cos(\theta)}{1} = \frac{x}{r} \quad \text{and} \quad \frac{\sin(\theta)}{1} = \frac{y}{r}.$$

Solving these equations for x and y, respectively, we obtain

$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta).$$

Looking back at Figure 9, we see that what we've found are the coordinates of the point T specified by the angle θ on the circumference of a circle of radius r. See the box below.

Suppose that the point $T = (x, y)$ is specified by the angle θ on the circumference of a circle of radius r. Then

$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta).$$

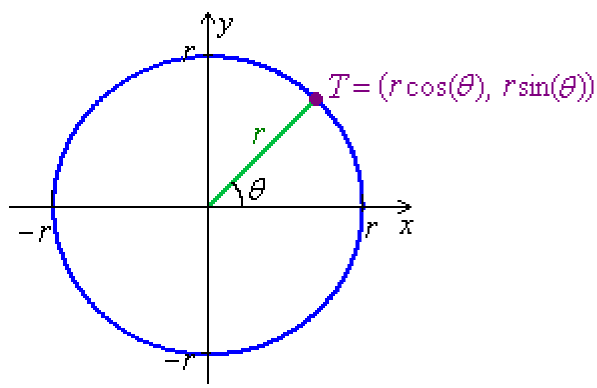

Figure 11: Circle of radius *r*.

EXAMPLE 2: A circle with a radius of 6 units is given in Figure 12. The point Q is specified by the angle α. Use the sine and cosine function to express the exact coordinates of point Q.

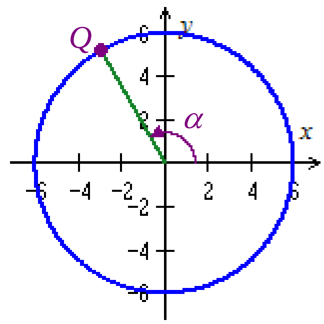

Figure 12

SOLUTION:

The point Q is specified by α on the circumference of a circle of radius 6 units. Thus,

$$Q = \left(6\cos(\alpha),\, 6\sin(\alpha)\right)$$

It turns out that the algebraic equation of a unit circle centered at the origin is $x^2 + y^2 = 1$, i.e., any point $(x,\, y)$ on the circumference of a unit circle centered at the origin satisfies the equation $x^2 + y^2 = 1$. Recall that the cosine and sine functions represent the horizontal and vertical coordinates of a point on the circumference of a unit circle; see Figure 13.

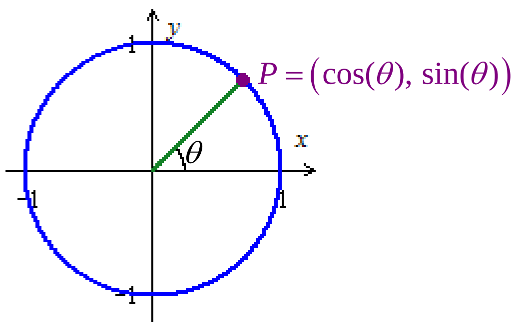

Figure 13

This means that ordered pairs of the form $(x,\, y) = \left(\cos(\theta),\, \sin(\theta)\right)$ satisfy the equation $x^2 + y^2 = 1$, i.e.,

$$\mathbf{\sin^2(\theta) + \cos^2(\theta) = 1}.$$

This identity is called the ***Pythagorean Identity***. Notice the "special notation" that we've employed to express the exponents for the trigonometric functions in the identity. Instead of using parentheses around the entire expressions, we can put the exponent between the letters that name the function and the input for the trig function. Thus, we can write an expression like "$\left(\sin(\theta)\right)^2$" as "$\sin^2(\theta)$".

EXAMPLE 3: If $\sin(\alpha) = \frac{1}{3}$ and $\frac{\pi}{2} < \alpha < \pi$ (i.e., α is in Quadrant II), find $\cos(\alpha)$.

SOLUTION:

Since the Pythagorean Theorem gives us an equation involving sine and cosine, we can use it to find one of the values when we know the other value. In this case, we know the value of $\sin(\alpha)$, so we can use the Pythagorean Theorem to find $\cos(\alpha)$:

$$\begin{aligned}
& \sin^2(\alpha) + \cos^2(\alpha) = 1 \\
\Rightarrow\quad & \left(\tfrac{1}{3}\right)^2 + \cos^2(\alpha) = 1 \\
\Rightarrow\quad & \cos^2(\alpha) = 1 - \left(\tfrac{1}{3}\right)^2 \\
\Rightarrow\quad & \cos(\alpha) = -\sqrt{\tfrac{8}{9}} \qquad \text{(we choose the negative square root since cosine is negative in Quadrant II.)} \\
\Rightarrow\quad & \cos(\alpha) = -\frac{2\sqrt{2}}{3}
\end{aligned}$$