Section I: Periodic Functions and Trigonometry

Chapter 2: Introduction to Periodic Functions

DEFINITION: A function f is ***periodic*** if its values repeat on regular intervals. Hence, f is periodic if there exists some constant c such that

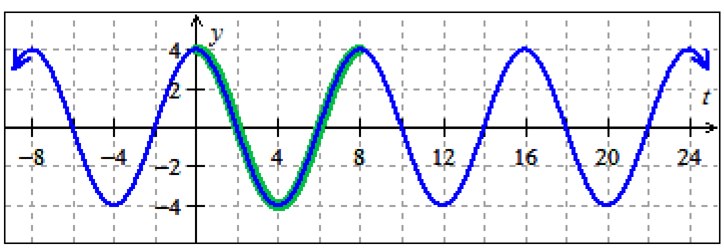
$$f(t+c) = f(t)$$

for all t in the domain of f such that $f(t+c)$ is defined. (Recall that this means that if the graph of $y = f(t)$ is shifted horizontally c units then it will appear unaffected.)

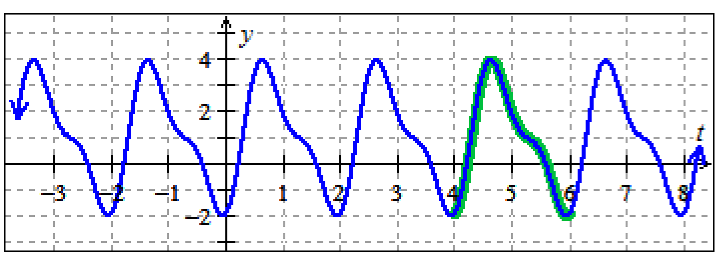
Any activity that repeats on a regular time interval can be described as *periodic*. For example, if the bell at a local church rings once every-hour-on-the-hour, then the function that relates the time of day to whether or not the bell will ring is a *periodic* function. Similarly if you take your dogs on a one-hour walk every day at 10 am, then the function that associates the time of day with whether or not you're on a walk with your dogs is a *periodic* function.

EXAMPLE 1: The following are graphs of periodic functions. (We know that they are periodic since an interval of each graph repeats over-and-over-and-over; that interval has been highlighted green in the graphs below.)

a.



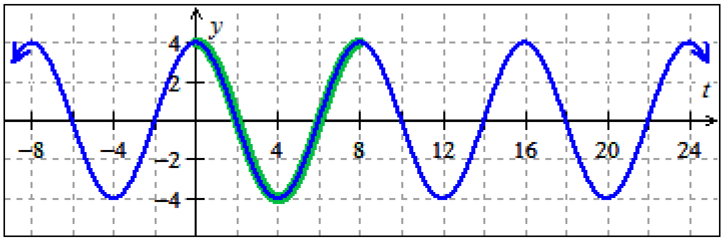
b.



DEFINITION: The ***period*** of a periodic function f is the smallest value $|c|$ such that $f(t+c) = f(t)$ for all t in the domain of f such that $f(t+c)$ is defined.

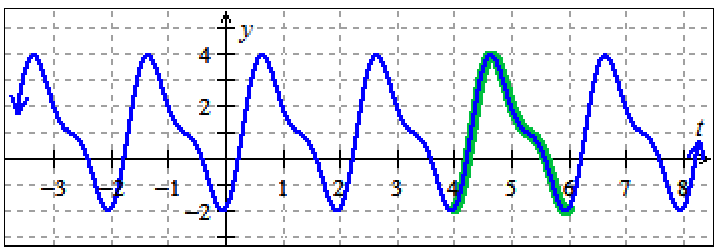
EXAMPLE 2: Find the period of the functions graphed below.

a.



The period of this function is 8 units since we can shift the graph horizontally 8 units, the graph will appear unaffected. (Notice that the "green interval" represents one period and is 8 units long.)

b.



The period of this function is 2 units since we can shift the graph horizontally 2 units, the graph will appear unaffected. (Notice that the "green interval" represents one period and is 2 units long.)

DEFINITIONS:

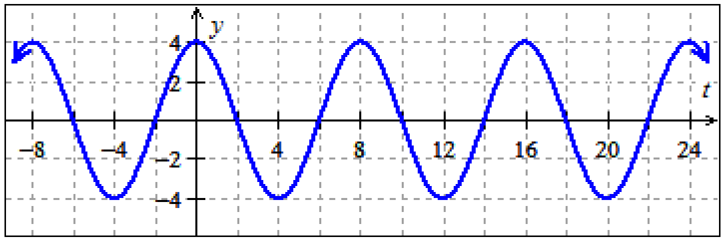
- The ***midline*** of a periodic function is the horizontal line midway between the function's minimum and maximum values.

  If $y = f(t)$ is periodic and f_{max} and f_{min} are the maximum and minimum values of f, respectively, then the equation of the midline is $y = \dfrac{f_{\text{max}} + f_{\text{min}}}{2}$.

- The ***amplitude*** of a periodic function is the distance between the function's maximum value and the midline (or the function's minimum value and the midline).

EXAMPLE 3: Find the midline and the amplitude of the functions graphed below.

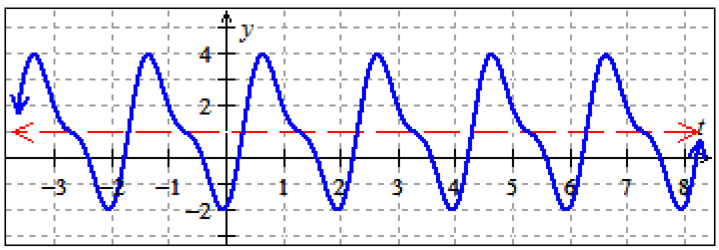
a.



The ***midline*** of this function is the t-axis (i.e., the line $y = 0$) since the maximum output for the function is 4 while the minimum output is –4, and $\frac{4+(-4)}{2} = 0$.

The ***amplitude*** of this function is 4 units.

b.



The ***midline*** of this function is the line $y = 1$ since the maximum output for the function is 4 while the minimum output is –2, and $\frac{4+(-2)}{2} = 1$.

The ***amplitude*** of this function is 3 units.

EXAMPLE 4: The Amusement Park has a Ferris wheel 200 feet in diameter. The wheel rotates at a constant rate and completes a rotation once every 40 minutes. Let $h(t)$ represent the height in feet of a Ferris wheel passenger t minutes after boarding the wheel at ground level. Sketch a graph of $y = h(t)$.

SOLUTION:

Since the Ferris wheel completes a rotation once every 40 minutes, the values of the height function $y = h(t)$ will repeat every 40 minutes so the period of $y = h(t)$ is 40 minutes.

Since the wheel is rotating at a constant rate, if it takes 40 minutes to complete a full rotation, it will take 20 minutes to travel half-way around the wheel. So at $t = 0$, the passenger will be at ground level (i.e., $h(0) = 0$); then at $t = 20$ the passenger will be at the top of the wheel (i.e., $h(20) = 200$), then at $t = 40$ the passenger will be at ground level again (i.e., $h(40) = 0$), etc. Furthermore, since it takes 20 minutes to travel from ground level to the top of the wheel, we can infer that it takes 10 minutes to travel half the distance from the ground to the top of the wheel, or 100 feet, so at $h(10) = 100$. Similarly, it takes 10 minutes to travel from the top of the wheel half down to a height of 100 feet, so $h(30) = 100$.

We can summarize this information in the table below, and then plot the ordered pairs $(t, h(t))$ on the coordinate plane in Figure 1. (Note that the colors in the table correspond to the colors of the points in Figure 1.)

t (minutes)	0	10	20	30	40	50	60	70	80
$h(t)$ (feet)	0	100	200	100	0	100	200	100	0

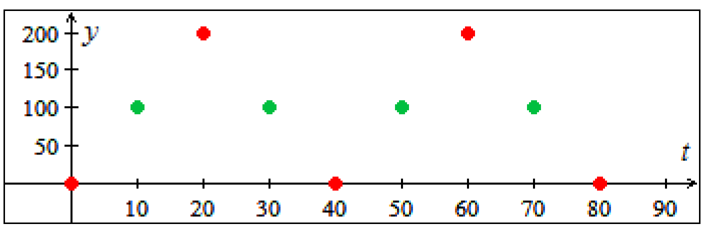


Figure 1: Some points on the graph of $y = h(t)$.

Now we need to connect the dots. To determine how the dots should be connected, let's imagine that we are the Ferris wheel's passengers. (To help you get a good mental image of the trip around the wheel, just imagine traveling around a circle; see Figure 2, below.)

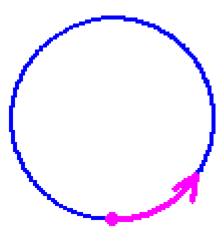


Figure 2: Simplified Ferris wheel

When we first begin to travel around the wheel (starting at ground level, i.e., at the 6 o'clock position on the wheel), at first we don't gain much elevation. After a short period (near the tip of the red arrow in Figure 2), we begin to gain elevation more and more quickly. One-quarter of the way around the wheel we'll be gaining elevation most quickly (since the wheel is vertical here). As we near the top of the wheel it gets flatter and flatter

so we'll begin to gain less and less elevation until we reach the top of the wheel. This tells us that our graph of $y = h(t)$ should be steep near the green dots less and less steep as it approaches the red dots. Let's use this information to connect the dots on our graph:

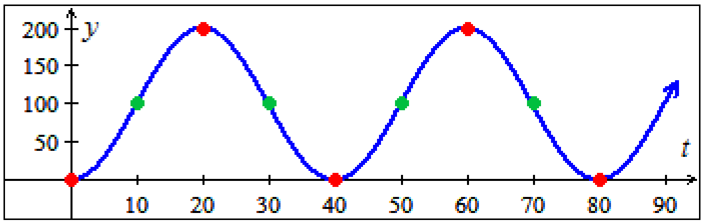


Figure 3: The graph of $y = h(t)$.