Section I: Periodic Functions and Trigonometry

Chapter 0: Sets and Numbers

DEFINITION: A **set** is a collection of objects specified in a manner that enables one to determine if a given object is or is not in the set.

In other words, a set is a well-defined collection of objects.

EXAMPLE: Which of the following represent a set?

a. The students registered for MTH 112 at PCC this quarter.

b. The good students registered for MTH 112 at PCC this quarter.

SOLUTION:

a. This represents a set since it is "well defined": We all know what it means to be registered for a class.

b. This does NOT represent a set since it is not well defined: There are many different understandings of what it means to be a good student (get an A or pass the class or attend class or avoid falling asleep in class).

EXAMPLE: Which of the following represent a set?

a. All of the really big numbers. **b.** All the whole numbers between 3 and 10.

SOLUTION:

a. It should be obvious why this does NOT represent a set. (What does it mean to be a "big number"?)

b. This represents a set. We can represent sets like **b** in **roster notation** by using "curly brackets".

"All of the whole numbers between 3 and 10" $= \{4,\ 5,\ 6,\ 7,\ 8,\ 9\}$

Roster Notation involves listing the elements in a set within *curly brackets*: "{ }".

DEFINITION: An object in a set is called an **element** of the set. (symbol: "$\in$")

EXAMPLE: 5 is an element of the set $\{4, 5, 6, 7, 8, 9\}$. We can express this symbolically:

$$5 \in \{4, 5, 6, 7, 8, 9\}$$

DEFINITION: Two sets are considered **equal** if they have the same elements.

We used this definition earlier when we wrote:

"All of the whole numbers between 3 and 10" $= \{4, 5, 6, 7, 8, 9\}$.

DEFINITION: A set S is a **subset** of a set T, denoted $S \subseteq T$, if all elements of S are also elements of T.

If S and T are sets and $S = T$, then $S \subseteq T$. Sometimes it is useful to consider a subset S of a set T that is not equal to T. In such a case, we write $S \subset T$ and say that S is a **proper subset** of T.

EXAMPLE: $\{4, 7, 8\}$ is a subset of the set $\{4, 5, 6, 7, 8, 9\}$.

We can express this fact symbolically by $\{4, 7, 8\} \subseteq \{4, 5, 6, 7, 8, 9\}$.

Since these two sets are not equal, $\{4, 7, 8\}$ is a *proper* subset of $\{4, 5, 6, 7, 8, 9\}$, so we can write

$$\{4, 7, 8\} \subset \{4, 5, 6, 7, 8, 9\}.$$

DEFINITION: The **empty set**, denoted Ø, is the set with no elements.

$$\varnothing = \{\ \ \}$$ There are NO elements in Ø.

The empty set is a subset of all sets. Note that $0 \neq \varnothing$.

DEFINITION: The **union** of two sets A and B, denoted $A \cup B$, is the set containing all of the elements in either A or B (or both A and B).

EXAMPLE: Consider the sets $\{4, 7, 8\}$, $\{0, 2, 4, 6, 8\}$, and $\{1, 3, 5, 7\}$. Then…

a. $\{4, 7, 8\} \cup \{1, 3, 5, 7\} = \{1, 3, 4, 5, 7, 8\}$

b. $\{4, 7, 8\} \cup \{0, 2, 4, 6, 8\} = \{0, 2, 4, 6, 7, 8\}$

c. $\{0, 2, 4, 6, 8\} \cup \{1, 3, 5, 7\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

DEFINITION: The **intersection** of two sets A and B, denoted $A \cap B$, is the set containing all of the elements in both A and B.

EXAMPLE: Consider the sets $\{4, 7, 8\}$, $\{0, 2, 4, 6, 8\}$, and $\{1, 3, 5, 7\}$. Then…

a. $\{4, 7, 8\} \cap \{0, 2, 4, 6, 8\} = \{4, 8\}$

b. $\{4, 7, 8\} \cap \{1, 3, 5, 7\} = \{7\}$

c. $\{0, 2, 4, 6, 8\} \cap \{1, 3, 5, 7\} = \varnothing$ These sets have no elements in common, so their intersection is the empty set.

EXAMPLE: All of the whole numbers (positive and negative) form a set. This set is called the **integers**, and is represented by the symbol $\mathbb{Z}$. We can express the set of integers in roster notation:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Note that $\mathbb{Z}$ is used to represent the integers because the German word for "number" is "zahlen."

Now that we have the integers, we can represent sets like "All of the whole numbers between 3 and 10" using **set-builder notation**:

SET-BUILDER NOTATION:

$$\text{"All the whole numbers between 3 and 10"} = \{x \mid x \in \mathbb{Z} \text{ and } 3 < x < 10\}$$

This vertical line means "such that"

Armed with set-builder notation, we can define important **sets of numbers**:

DEFINITIONS: The set of **natural numbers**: $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$

The set of **integers**: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

The set of **rational numbers** (i.e., the set of fractions):

$$\mathbb{Q} = \left\{x \,\middle|\, x = \tfrac{p}{q} \text{ and } p,\, q \in \mathbb{Z} \text{ and } q \neq 0\right\}$$

The set of **real numbers**: $\mathbb{R}$ (All the numbers on the number line.)

The set of **complex numbers**:

$$\mathbb{C} = \left\{x \,\middle|\, x = a + bi \text{ and } a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}\right\}$$

Note that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$, i.e., the set of natural numbers is a subset of the set of integers which is a subset of the set of rational numbers which is a subset of the real numbers which is a subset of the set of complex numbers.

Throughout this course, we will assume that the number-set in question is the real numbers, $\mathbb{R}$, unless we are specifically asked to consider an alternative set.

Since we use the real numbers so often, we have special notation for subsets of the real numbers: **interval notation**. Interval notation involves square or round brackets. Use the examples below to understand how interval notation works.

EXAMPLE:

a. $\underbrace{\{x \mid x \in \mathbb{R} \text{ and } -2 \le x \le 3\}}_{\text{Set-builder Notation}} = \underbrace{[-2, 3]}_{\text{Interval Notation}}$

We use square brackets here since the endpoints are included

b. $\underbrace{\{x \mid x \in \mathbb{R} \text{ and } -2 < x < 3\}}_{\text{Set-builder Notation}} = \underbrace{(-2, 3)}_{\text{Interval Notation}}$

We use round brackets here since the endpoints are NOT included.

c. $\underbrace{\{x \mid x \in \mathbb{R} \text{ and } -2 < x \le 3\}}_{\text{Set-builder Notation}} = \underbrace{(-2, 3]}_{\text{Interval Notation}}$

We use a round bracket on the left since -2 is NOT included.

d. $\underbrace{\{x \mid x \in \mathbb{R} \text{ and } -2 \le x < 3\}}_{\text{Set-builder Notation}} = \underbrace{[-2, 3)}_{\text{Interval Notation}}$

We use a round bracket on the right since 3 is NOT included.

EXAMPLE: When the interval has no upper (or lower) bound, the symbol ∞ (or $-\infty$) is used.

a. $\underbrace{\{x \mid x \in \mathbb{R} \text{ and } x \le 4\}}_{\text{Set-builder Notation}} = \underbrace{(-\infty, 4]}_{\text{Interval Notation}}$

We ALWAYS use a round bracket with $-\infty$ since it is NOT a number in the set.

b. $\underbrace{\{x \mid x \in \mathbb{R} \text{ and } x \ge 4\}}_{\text{Set-builder Notation}} = \underbrace{[4, \infty)}_{\text{Interval Notation}}$

We ALWAYS use a round bracket with ∞ since it is NOT a number in the set.

EXAMPLE: Simplify the following expressions.

a. $(-4, \infty) \cup [-8, 3]$

b. $(-4, \infty) \cup (-\infty, 2]$

c. $(-4, \infty) \cap (-\infty, 2]$

d. $(-4, \infty) \cap [-10, -5]$

SOLUTION:

a. $(-4, \infty) \cup [-8, 3] = [-8, \infty)$

b. $(-4, \infty) \cup (-\infty, 2] = (-\infty, \infty) = \mathbb{R}$

c. $(-4, \infty) \cap (-\infty, 2] = (-4, 2]$

d. $(-4, \infty) \cap [-10, -5] = \varnothing$