Section IV: Vectors

Chapter 2: The Dot Product

In the previous chapter we studied how to add and subtract vectors and how to multiply vectors by scalars. In this chapter we will study how to multiply one vector by another using an operation called the **dot product**. Since we are focusing on two-dimensional vectors in this course, we will define the dot product in terms of two-dimensional vectors:

DEFINITION: If $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$, then **the dot product of $\vec{u}$ and $\vec{v}$**, denoted $\vec{u} \cdot \vec{v}$, is defined as follows:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 .$$

Thus, to compute the dot product of two vectors, we simply multiply the horizontal components of the two vectors and the vertical components of the two vectors and then add the results. It is important to note that the dot product produces a **scalar**.

EXAMPLE 1: If $\vec{a} = \langle 3, -9 \rangle$ and $\vec{b} = \langle 6, -1 \rangle$, find $\vec{a} \cdot \vec{b}$.

SOLUTION:

$$\begin{aligned} \vec{a} \cdot \vec{b} &= \langle 3, -9 \rangle \cdot \langle 6, -1 \rangle \\ &= 3 \cdot 6 + (-9) \cdot (-1) \\ &= 18 + 9 \\ &= 27 \end{aligned}$$

Notice that the result is a scalar, not a vector.

Now let's state some properties of the dot product. Afterwards, we will justify each of the properties.

Properties of the Dot Product

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors then the following properties hold true:

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (commutative property)

2. $\vec{u} \cdot \left(\vec{v} + \vec{w}\right) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (distributive property)

3. $\vec{v} \cdot \vec{v} = \left\|\vec{v}\right\|^2$

4. $\vec{u} \cdot \vec{v} = \left\|\vec{u}\right\| \cdot \left\|\vec{v}\right\| \cos(\theta)$ where θ is the angle between $\vec{u}$ and $\vec{v}$.

The fourth property in our list is probably the most interesting since it suggests that the dot product can be used to measure the alignment of two vectors. Before discussing this property, let's justify the other properties in our list.

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1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (commutative property)

To justify this statement, let's compute $\vec{u} \cdot \vec{v}$ for some generic vectors $\vec{u}$ and $\vec{v}$ and show that the result is equal to $\vec{v} \cdot \vec{u}$.

Let $\vec{u} = \left\langle u_1, u_2 \right\rangle$ and $\vec{v} = \left\langle v_1, v_2 \right\rangle$. Then

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \left\langle u_1, u_2 \right\rangle \cdot \left\langle v_1, v_2 \right\rangle \\ &= u_1 v_1 + u_2 v_2 \\ &= v_1 u_1 + v_2 u_2 \quad \text{(since scalar multiplication is commutative)} \\ &= \vec{v} \cdot \vec{u} \end{aligned}$$

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2. $\vec{u}\cdot\left(\vec{v}+\vec{w}\right)=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$ (distributive property)

To justify this statement, let's compute $\vec{u}\cdot\left(\vec{v}+\vec{w}\right)$ for some generic vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ and show that the result is equal to $\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$.

Let $\vec{u}=\langle u_1, u_2\rangle$, $\vec{v}=\langle v_1, v_2\rangle$, and $\vec{w}=\langle w_1, w_2\rangle$. Then

$$
\begin{aligned}
\vec{u}\cdot\left(\vec{v}+\vec{w}\right) &= \langle u_1, u_2\rangle\cdot\left(\langle v_1, v_2\rangle+\langle w_1, w_2\rangle\right)\\
&= \langle u_1, u_2\rangle\cdot\left(\langle v_1+w_1, v_2+w_2\rangle\right)\\
&= u_1(v_1+w_1)+u_2(v_2+w_2)\\
&= u_1v_1+u_1w_1+u_2v_2+u_2w_2 \quad \text{(since scalar multiplication is distributive)}\\
&= u_1v_1+u_2v_2+u_1w_1+u_2w_2 \quad \text{(since scalar addition is commutative)}\\
&= \langle u_1, u_2\rangle\cdot\langle v_1, v_2\rangle+\langle u_1, u_2\rangle\cdot\langle w_1, w_2\rangle\\
&= \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}
\end{aligned}
$$

3. $\vec{v}\cdot\vec{v}=\left\|\vec{v}\right\|^2$

To justify this statement, let's compute $\vec{v}\cdot\vec{v}$ for some generic vector $\vec{v}$ and show that the result is equal to $\left\|\vec{v}\right\|^2$.

Let $\vec{v}=\langle v_1, v_2\rangle$. Then

$$
\begin{aligned}
\vec{v}\cdot\vec{v} &= \langle v_1, v_2\rangle\cdot\langle v_1, v_2\rangle\\
&= v_1v_1+v_2v_2\\
&= {v_1}^2+{v_2}^2\\
&= \left\|\vec{v}\right\|^2 \quad \left(\text{since } \left\|\vec{v}\right\|=\sqrt{{v_1}^2+{v_2}^2}\right)
\end{aligned}
$$

4. $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos(\theta)$ where θ is the angle between $\vec{u}$ and $\vec{v}$.

To justify this statement, let's first draw two vectors $\vec{u}$ and $\vec{v}$ so that θ is the angle between the vectors; see Figure 1.

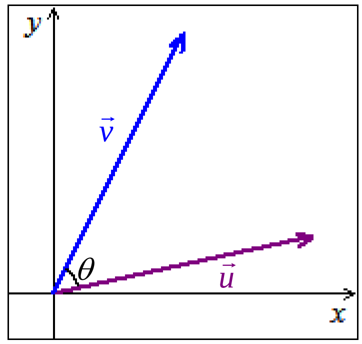

Figure 1

Now let's construct the vector $\vec{w} = \vec{v} - \vec{u}$. Recall from the previous chapter that we can obtain the vector $\vec{v} - \vec{u}$ by drawing an arrow that starts at the tip of $\vec{u}$ and ends at the tip of $\vec{v}$. We've drawn $\vec{w} = \vec{v} - \vec{u}$ in Figure 2.

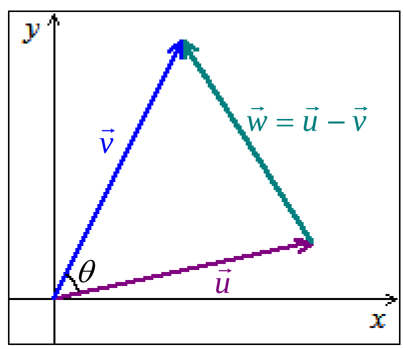

Figure 2

If we think of the arrows as being line segments instead of arrows, we have a triangle with side lengths $\|\vec{u}\|$, $\|\vec{v}\|$, and $\|\vec{w}\|$ and angle θ opposite the side $\|\vec{w}\|$; see Figure 3.

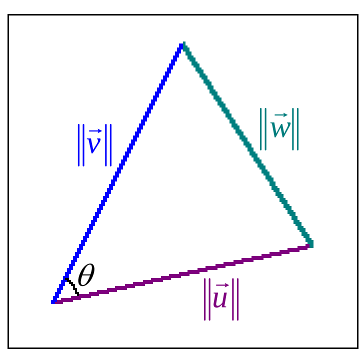

Figure 3

Now we can use the Law of Cosines to obtain an equation that relates the magnitudes of the vectors and angle θ.

$$\|\vec{w}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos(\theta) \qquad \langle 1\rangle$$

We can use this equation to obtain the statement given in fourth property in the table above. First, let's find $\|\vec{w}\|^2$. Recall that $\vec{w} = \vec{v} - \vec{u}$. So

$$\begin{aligned}
\|\vec{w}\|^2 &= \vec{w}\cdot\vec{w} && \text{(using the property } \mathbf{3} \text{ from the table above)}\\
&= (\vec{v} - \vec{u})\cdot(\vec{v} - \vec{u})\\
&= \vec{v}\cdot\vec{v} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{u} && \text{(using the property } \mathbf{2} \text{ from the table above)}\\
&= \vec{v}\cdot\vec{v} - \vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{u} && \text{(using the property } \mathbf{1} \text{ from the table above)}\\
&= \|\vec{v}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{u}\|^2 && \text{(using the property } \mathbf{3} \text{ from the table above)}
\end{aligned}$$

We can now substitute $\|\vec{v}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{u}\|^2$ for $\|\vec{w}\|^2$ in the equation labeled $\langle 1\rangle$, above, and simplify to obtain equation given in the forth property in the table:

$$\begin{aligned}
&\cancel{\|\vec{v}\|^2} - 2\vec{u}\cdot\vec{v} + \cancel{\|\vec{u}\|^2} = \cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} - 2\|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos(\theta)\\
\Rightarrow\quad &\cancel{-2}\vec{u}\cdot\vec{v} = \cancel{-2}\|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos(\theta)\\
\Rightarrow\quad &\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos(\theta)
\end{aligned}$$

We can use the fact that $\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cos(\theta)$ to see that the dot product is intimately related to the alignment of the vectors $\vec{u}$ and $\vec{v}$.

EXAMPLE 2: If the angle between $\vec{u}$ and $\vec{v}$ is $\theta = 90^\circ$ (i.e., if $\vec{u}$ and $\vec{v}$ are perpendicular), find $\vec{u} \cdot \vec{v}$.

SOLUTION:

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos(\theta) \\ &= \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\left(90^\circ\right) \\ &= \|\vec{u}\| \cdot \|\vec{v}\| \cdot 0 \\ &= 0 \end{aligned}$$

So if $\vec{u}$ and $\vec{v}$ are perpendicular, $\vec{u} \cdot \vec{v} = 0$.

EXAMPLE 3: If $\vec{u}$ and $\vec{v}$ are non-zero vectors and $\vec{u} \cdot \vec{v} > 0$, what can you say about the angle θ between vectors $\vec{u}$ and $\vec{v}$. What if $\vec{u} \cdot \vec{v} < 0$?

SOLUTION:

Recall that $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos(\theta)$. Since $\vec{u} > 0$ and $\vec{v} > 0$, we see that the sign of $\vec{u} \cdot \vec{v}$ depends on the sign of $\cos(\theta)$.

- If $\cos(\theta) > 0$ then $\vec{u} \cdot \vec{v} > 0$; since $\cos(\theta) > 0$ when $0 \le \theta < 90^\circ$, we can conclude that $\vec{u} \cdot \vec{v} > 0$ when the angle between the vectors is acute (i.e., less than 90°).

- If $\cos(\theta) < 0$ then $\vec{u} \cdot \vec{v} < 0$; since $\cos(\theta) < 0$ when $90^\circ < \theta \le 180^\circ$, we can conclude that $\vec{u} \cdot \vec{v} < 0$ when the angle between the vectors is obtuse (i.e., greater than 90°).

In the two examples above we see that the dot product can be used to learn about the alignment of two vectors. In fact, the dot product can be used to find the angle between two vectors; see Example 4, below.

EXAMPLE 4: Find the angle between the vectors $\vec{a} = \langle 3, -9 \rangle$ and $\vec{b} = \langle 6, -1 \rangle$ from Ex. 1.

SOLUTION:

To find the angle θ between the two vectors we will use the fact that $\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos(\theta)$. Recall from Example 1 that $\vec{a} \cdot \vec{b} = 27$. Let's find $\|\vec{a}\|$ and $\|\vec{b}\|$:

$$\begin{aligned} \|\vec{a}\| &= \sqrt{(3)^2 + (-9)^2} \\ &= \sqrt{9 + 81} \\ &= 3\sqrt{10} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \|\vec{b}\| &= \sqrt{(6)^2 + (-1)^2} \\ &= \sqrt{36 + 1} \\ &= \sqrt{37} \end{aligned}$$

Thus,

$$\begin{aligned} & \vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos(\theta) \\ \Rightarrow \quad & 27 = 3\sqrt{10} \cdot \sqrt{37} \cdot \cos(\theta) \\ \Rightarrow \quad & \cos(\theta) = \tfrac{27}{3\sqrt{10} \cdot \sqrt{37}} \\ \Rightarrow \quad & \theta = \cos^{-1}\left(\tfrac{9}{\sqrt{370}}\right) \approx 62.1^\circ \end{aligned}$$

So the angle between $\vec{a}$ and $\vec{b}$ is about 62.1°; see Figure 4.

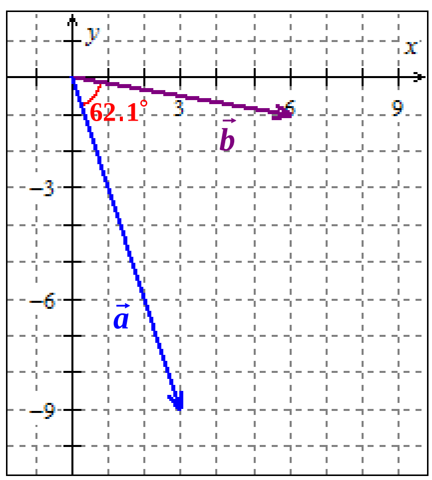

Figure 4