Section IV: Vectors

Chapter 1: Introduction to Vectors

Vectors are mathematical objects used to represent physical quantities like velocity, force, and displacement. Unlike ordinary numbers (or **scalars**), vectors describe *both* magnitude and direction. So, for example, we can describe the velocity (i.e., the speed *and* direction) of an object with a vector.

DEFINITION: A **vector** is a mathematical object used to represent a physical quantity that has both a *magnitude* (i.e., size) and a *direction*.

In order to distinguish between vectors from scalars (i.e., numbers) we need to use a different notation to denote vectors. In this course, we will use a small arrow above the vector name to denote a vector, so that $\vec{v}$ and $\vec{s}$ represent vectors while v and s represent scalars. (Note that our textbook uses bold text to represent vectors but this isn't possible for handwritten work; instead the "arrow notation" is necessary for handwritten work so I prefer to also use arrows in typed work.)

In this class we will focus on **two-dimensional vectors**. A two-dimensional vectors can be represented by an **arrow** on the coordinate plane. The **length** of the arrow represents the **magnitude** of the vector and the **direction** that the arrow points represents the direction of the vector. (We traditionally use the **angle between the positive x-axis and the arrow** to describe the **direction** of the vector.)

EXAMPLE 1: The vector $\vec{v}$ is depicted as an arrow on the coordinate plane in Figure 1.

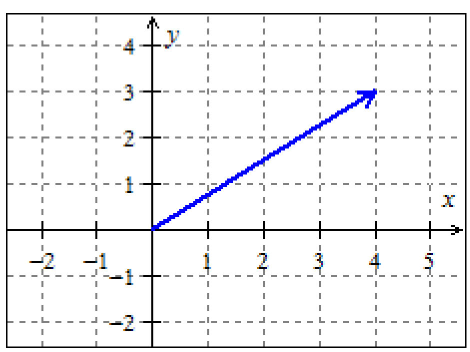

Figure 1: Arrow representing $\vec{v}$.

The **tip** of the vector is where the arrow ends and the **tail** of the vector is where the arrow begins. Thus, the tip of $\vec{v}$ is at the point $(4, 3)$ and the tail of the vector is at the origin, $(0, 0)$.

As mentioned above, the **length** of the arrow represents the **magnitude** of the vector. We denote the magnitude of vector $\vec{v}$ by $\|\vec{\boldsymbol{v}}\|$. To find the magnitude of $\vec{v}$, we need to find the length of the arrow; we can do this by thinking of the arrow as being the hypotenuse of a right-triangle with side lengths 4 and 3 (see Figure 2) and then use the Pythagorean Theorem to find $\|\vec{v}\|$.

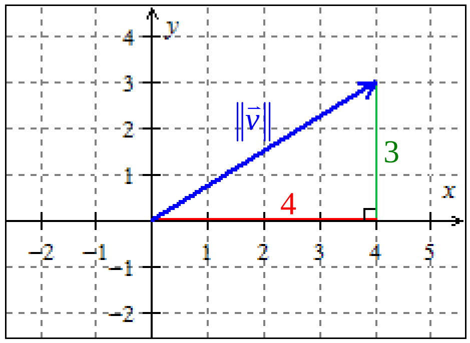

Figure 2

$$\begin{aligned} \|\vec{v}\|^2 &= 3^2 + 4^2 \\ \Rightarrow \|\vec{v}\|^2 &= 9 + 16 \\ \Rightarrow \|\vec{v}\|^2 &= 25 \\ \Rightarrow \quad \|\vec{v}\| &= 5 \end{aligned}$$

So the magnitude of $\vec{v}$ is 5 units.

We can find the angle between the positive x-axis and the arrow to describe the **direction** of the vector. We've denoted this angle by θ in Figure 3.

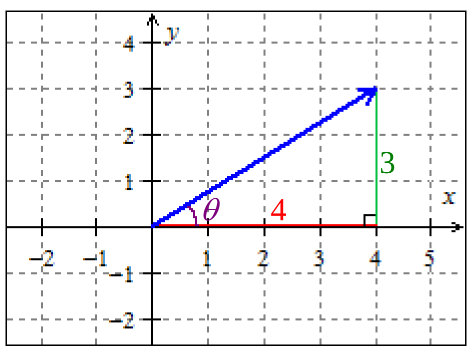

Figure 3: The components of $\vec{v}$.

We can use the trigonometry that we learned earlier in the course to find θ:

$$\begin{aligned} \tan(\theta) &= \tfrac{3}{4} \\ \Rightarrow \qquad \theta &= \tan^{-1}\left(\tfrac{3}{4}\right) \\ \Rightarrow \qquad \theta &\approx 36.87^\circ \end{aligned}$$

Although the magnitude and direction of the vector describe it completely, it is often useful to describe a vector by using its **horizontal and vertical components**. The *horizontal component* of $\vec{v}$ in Figure 3 (above) is 4 units and a *vertical component* of vector $\vec{v}$ is 3 units. Thus, we say that the **component form of vector $\vec{v}$** is $\langle 4, 3 \rangle$.

It is important to recognize that we could translate this vector anywhere in the coordinate plane and it would still be the same vector. For example, all of the arrows in Figure 4 represent $\vec{v}$ since all of these vectors have a horizontal component of 4 and a vertical component of 3.

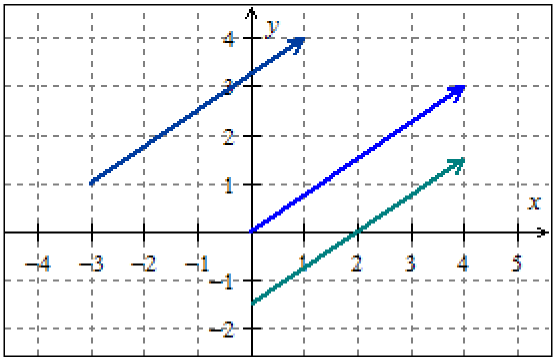

Figure 4: Three copies of $\vec{v}$.

EXAMPLE 2: Find the component form of the vector $\vec{s}$ given in Figure 5 below.

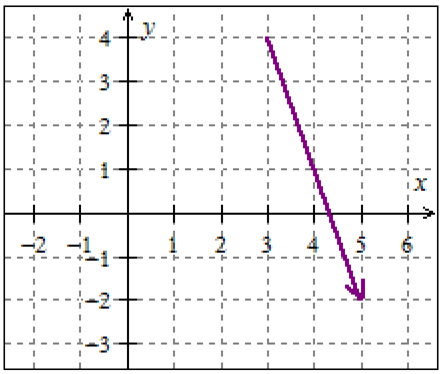

Figure 5: $\vec{s}$.

SOLUTION:

To find the *horizontal component* of $\vec{s}$ we need to determine the *horizontal distance* between the tip and the tail of the vector's arrow, and to find the *vertical component* of $\vec{s}$, we need to determine the *vertical distance* between the tip and the tail of the vector's arrow. As we can see in Figure 6 (below), the *horizontal component* of $\vec{s}$ is 2 units and the *vertical component* of $\vec{s}$ is –6 units. Note that the vertical component is negative since the arrow travels *down 6 units*, or *vertically –6 units*. The component form of $\vec{s}$ is $\langle 2, -6 \rangle$.

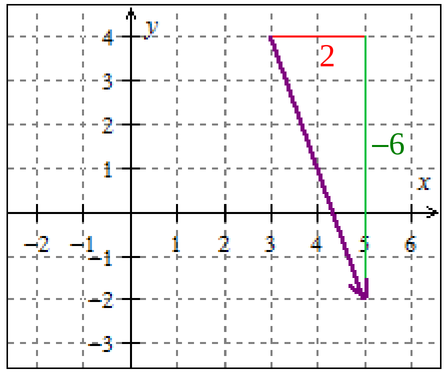

Figure 6: The components of $\vec{s}$.

If we translate vector $\vec{s}$ so that its tail is at the origin, we see that its tip is at the point $(2, -6)$; see Figure 7.

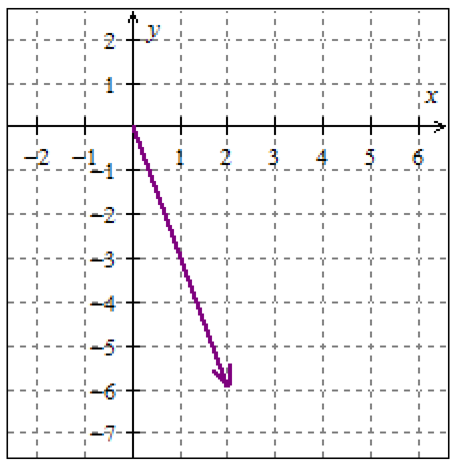

Figure 7: $\vec{s}$ translated so that its tail is at the origin.

Notice that in Example 1, the tail of $\vec{v}$ is at the origin and its tip is at the point $(4, 3)$, and the component form of $\vec{v}$ is $\langle 4, 3 \rangle$.) In general, a vector with component form $\langle a, b \rangle$ can be represented by an arrow on the coordinate plane whose tail is at the origin and whose tip is at the point (a, b).

EXAMPLE 3: Find the component form of the vector $\vec{r}$ given in Figure 8.

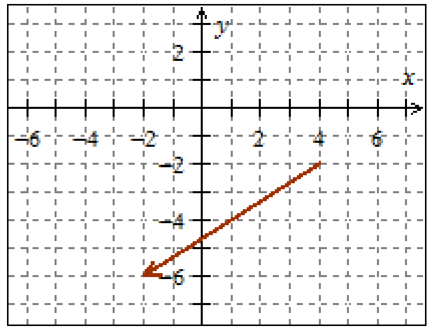

Figure 8: $\vec{r}$.

SOLUTION:

As we can see in Figure 9, the *horizontal component* of $\vec{r}$ is -6 units and the *vertical component* of $\vec{r}$ is -4 units so the component form of $\vec{r}$ is $\langle -6, -4 \rangle$.

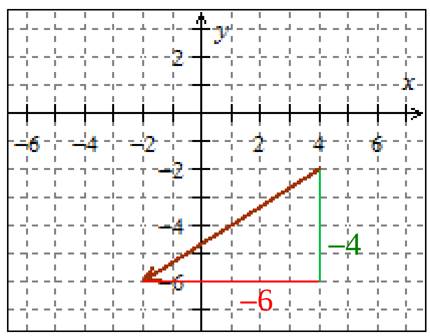

Figure 9: The components of $\vec{r}$.

We can translate $\vec{r} = \langle -6, -4 \rangle$ so that its tail is at the origin (see Figure 10); its tip is at the point $(-6, -4)$ which agrees with what we noticed above.

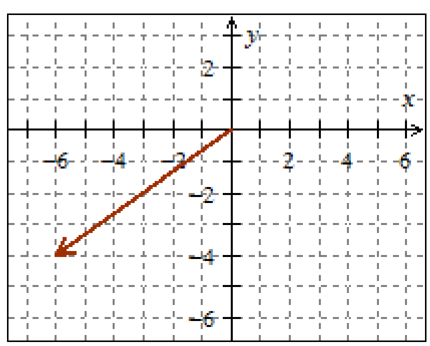

Figure 10: $\vec{r}$ translated so that its tail is at the origin.

Vectors Operations

We can multiply any vector by a scalar (i.e., a number) and we can add or subtract any two vectors.

When we **multiply a vector by a scalar**, we simply multiply the respective components of the vector by the scalar. Thus, if $\vec{a} = \langle a_1, a_2 \rangle$ and $k \in \mathbb{R}$, then $k\vec{a} = \langle ka_1, ka_2 \rangle$.

EXAMPLE 4: Let $\vec{v} = \langle 4, 3 \rangle$ (from Example 1).

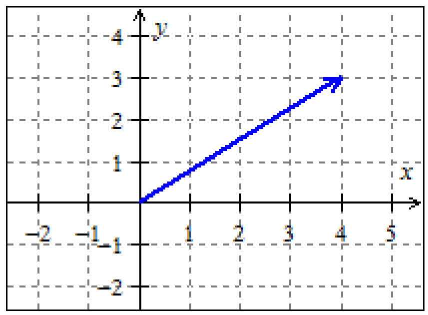

Figure 11: $\vec{v}$.

a. Find $2\vec{v}$. **b.** Find $\frac{1}{2}\vec{v}$. **c.** Find $-\vec{v}$.

SOLUTION:

a. $$\begin{aligned} 2\vec{v} &= 2 \cdot \langle 4, 3 \rangle \\ &= \langle 2 \cdot 4, 2 \cdot 3 \rangle. \\ &= \langle 8, 6 \rangle \end{aligned}$$

In Figure 12 we've drawn an arrow representing $2\vec{v}$. Notice that $2\vec{v}$ is **twice** as long as $\vec{v}$ yet it points **in the same direction**.

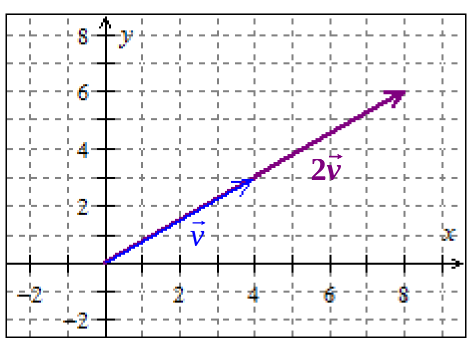

Figure 12: $\vec{v}$ and $2\vec{v}$.

b. $\frac{1}{2}\vec{v} = \frac{1}{2}\cdot\langle 4,\, 3\rangle$

$$= \left\langle \tfrac{1}{2}\cdot 4,\, \tfrac{1}{2}\cdot 3\right\rangle .$$

$$= \left\langle 2,\, \tfrac{3}{2}\right\rangle$$

In Figure 13 we've drawn an arrow representing $\frac{1}{2}\vec{v}$. Notice that $\frac{1}{2}\vec{v}$ is **half** as long as $\vec{v}$ yet it **points in the same direction**.

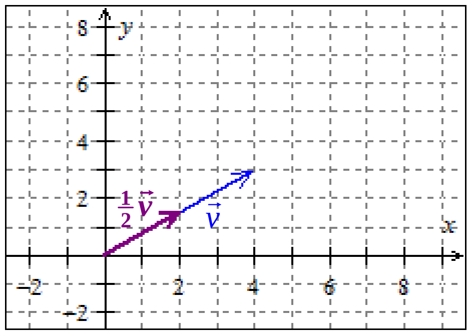

Figure 13: $\vec{v}$ and $\frac{1}{2}\vec{v}$.

c. $-\vec{v} = -1\cdot\vec{v}$

$$= -1\cdot\langle 4,\, 3\rangle$$

$$= \langle -1\cdot 4,\, -1\cdot 3\rangle .$$

$$= \langle -4,\, -3\rangle$$

In Figure 14 we've drawn an arrow representing $-\vec{v}$. Notice that $-\vec{v}$ is the **same** length as $\vec{v}$ yet it points **in the opposite direction**.

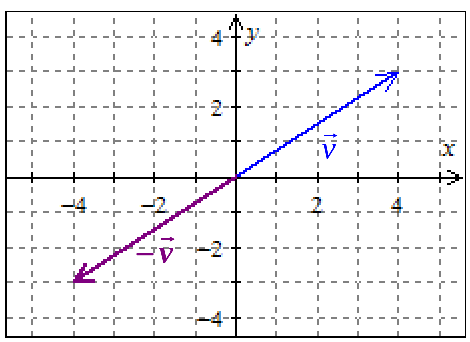

Figure 14: $\vec{v}$ and $-\vec{v}$.

If $\vec{a} = \langle a_1,\, a_2\rangle$ is a vector and $k \in \mathbb{R}$ then $k\vec{a} = \langle ka_1,\, ka_2\rangle$ has magnitude $|k|\cdot\|\vec{a}\|$. If $k > 0$ then $k\vec{a}$ points in the same direction as $\vec{a}$; if $k < 0$ then $k\vec{a}$ points in the opposite direction as $\vec{a}$.

When we **add or subtract vectors**, we simply add the respective components of the vectors. Thus, if $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$, then $\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$ and $\vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$

EXAMPLE 5: Let $\vec{v} = \langle 4, 3 \rangle$ (from Example 1) and $\vec{s} = \langle 2, -6 \rangle$ (from Example 2). Find $\vec{v} + \vec{s}$ and $\vec{v} - \vec{s}$.

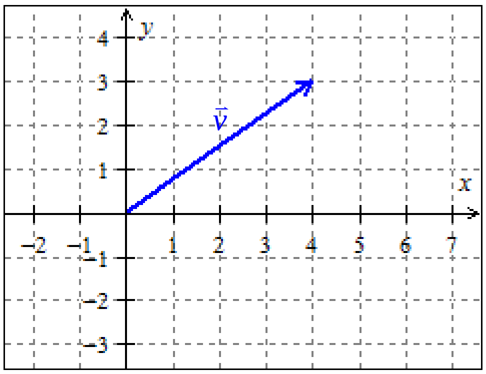

Figure 15a: $\vec{v}$.

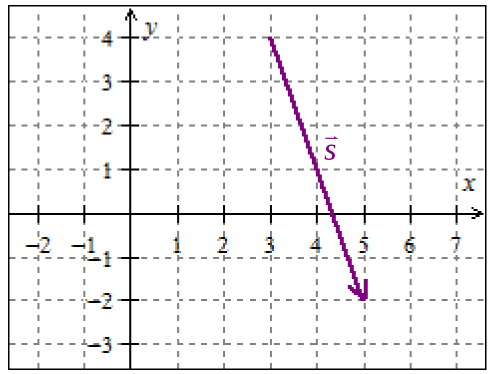

Figure 15b: $\vec{s}$.

SOLUTION:

Let's start by finding $\vec{v} + \vec{s}$:

$$\begin{aligned}\vec{v} + \vec{s} &= \langle 4, 3 \rangle + \langle 2, -6 \rangle \\ &= \langle 4 + 2, 3 + (-6) \rangle \\ &= \langle 6, -3 \rangle.\end{aligned}$$

We can also add vectors by using arrows on the coordinate plane by connecting the tip of the first arrow to the tail of the second arrow. In Figure 16 we show $\vec{v} + \vec{s}$:

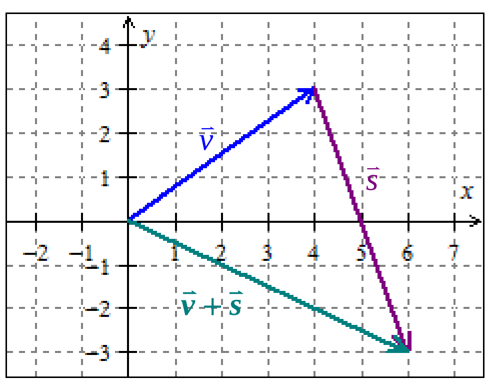

Figure 16: $\vec{v} + \vec{s}$.

Notice that the arrow for $\vec{v} + \vec{s} = \langle 6, -3 \rangle$ starts at the origin and ends at the point $(6, -3)$ which should give us confidence that these two ways of adding vectors (using components and using arrows) are equivalent.

Now, let's find $\vec{v} - \vec{s}$:

$$\begin{aligned}\vec{v} - \vec{s} &= \langle 4, 3\rangle - \langle 2, -6\rangle \\ &= \langle 4-2, 3-(-6)\rangle \\ &= \langle 2, 9\rangle\end{aligned}$$

We can also subtract vectors by using arrows on the coordinate plane. Notice that

$$\vec{v} - \vec{s} = \vec{v} + (-\vec{s})$$

Thus, we can obtain $\vec{v} - \vec{s}$ by adding $\vec{v}$ and $-\vec{s}$, i.e., by connecting the tip $\vec{v}$ with the tail of $-\vec{s}$; see Figure 17.

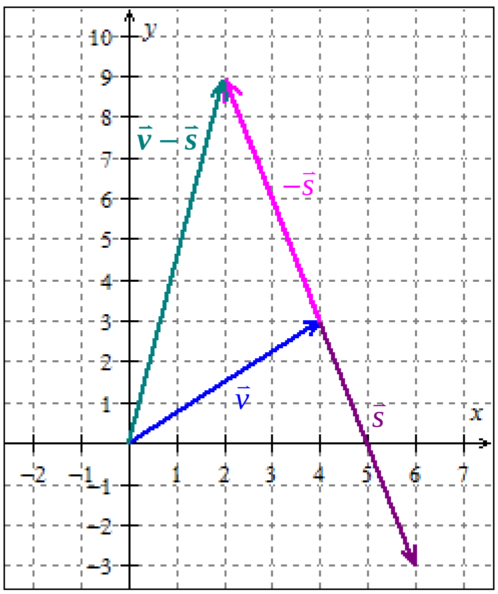

Figure 17

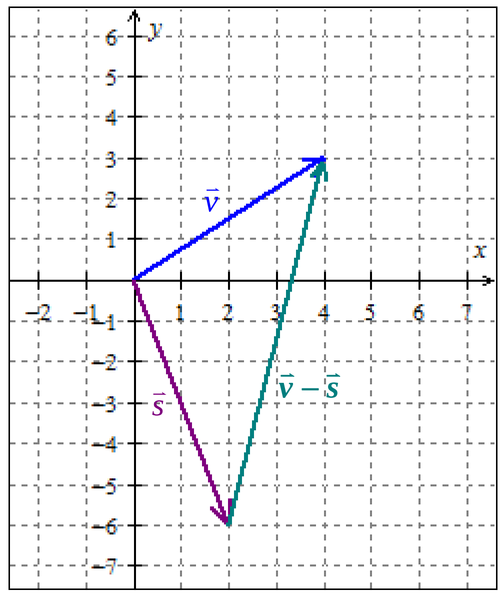

Figure 18

Notice that the arrow for $\vec{v} - \vec{s} = \langle 2, 9\rangle$ starts at the origin and ends at the point $(2, 9)$ which should give us confidence that these two ways of subtracting vectors (using components and using arrows) are equivalent. Notice that if we start both $\vec{v}$ and $\vec{s}$ at the origin, then $\vec{v} - \vec{s}$ is equivalent to the vector that starts at the tip of $\vec{s}$ and ends at the tip of $\vec{v}$; see Figure 18.

In order to facilitate the communication and manipulation of vectors, it is useful to consider **unit vectors**.

DEFINITION: A **unit vector** is a vector whose magnitude is 1 unit. So if $\vec{a}$ is a unit vector then $\|\vec{a}\| = 1$.

The **standard unit vectors** are the unit vectors that point in the horizontal and vertical directions.

DEFINITION:

The vector $\vec{i}$ is the unit vector that points in the **positive horizontal direction**. Since its horizontal component is 1 and its vertical component is 0, we see that $\vec{i} = \langle 1, 0 \rangle$.

The vector $\vec{j}$ is the unit vector that points in the **positive vertical direction**. Since its horizontal component is 0 and its vertical component is 1, we see that $\vec{j} = \langle 0, 1 \rangle$.

Since $\vec{i}$ and $\vec{j}$ are unit vectors, $\|\vec{i}\| = 1$ and $\|\vec{j}\| = 1$.

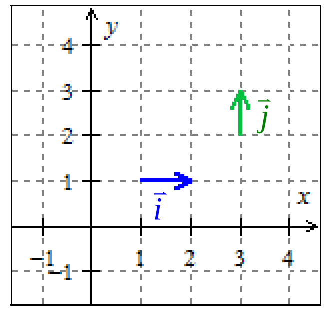

Figure 19: Unit vectors $\vec{i}$ and $\vec{j}$.

We can use vectors $\vec{i}$ and $\vec{j}$ to describe all other two-dimensional vectors. For example, in order to describe $\vec{v} = \langle 4, 3 \rangle$ (from Example 1) we can use vectors $\vec{i}$ and $\vec{j}$:

$$\begin{aligned} \vec{v} &= \langle 4, 3 \rangle \\ &= \langle 4, 0 \rangle + \langle 0, 3 \rangle \\ &= 4 \cdot \langle 1, 0 \rangle + 3 \cdot \langle 0, 1 \rangle \\ &= 4\vec{i} + 3\vec{j} \end{aligned}$$

Similarly, we can represent $\vec{s} = \langle 2, -6 \rangle$ and $\vec{r} = \langle -6, -4 \rangle$ (from the examples above) using vectors $\vec{i}$ and $\vec{j}$:

$$\begin{aligned} \vec{s} &= \langle 2, -6 \rangle \\ &= \langle 2, 0 \rangle + \langle 0, -6 \rangle \\ &= 2 \cdot \langle 1, 0 \rangle + (-6) \cdot \langle 0, 1 \rangle \\ &= 2\vec{i} + \left(-6\vec{j}\right) \\ &= 2\vec{i} - 6\vec{j} \end{aligned}$$

and

$$\begin{aligned} \vec{r} &= \langle -6, -4 \rangle \\ &= \langle -6, 0 \rangle + \langle 0, -4 \rangle \\ &= -6 \cdot \langle 1, 0 \rangle + (-4) \cdot \langle 0, 1 \rangle \\ &= -6\vec{i} + \left(-4\vec{j}\right) \\ &= -6\vec{i} - 4\vec{j} \end{aligned}$$

In general, if $\vec{a} = \langle a_1, a_2 \rangle$ is a vector, then $\vec{a} = a_1\vec{i} + a_2\vec{j}$.

EXAMPLE 6: Find the magnitude and direction of the vector $\vec{p} = -3\vec{i} + 7\vec{j}$.

SOLUTION:

First, notice that we can write $\vec{p}$ in component form as $\langle -3, 7 \rangle$. We've drawn an arrow representing $\vec{p}$ in Figure 20.

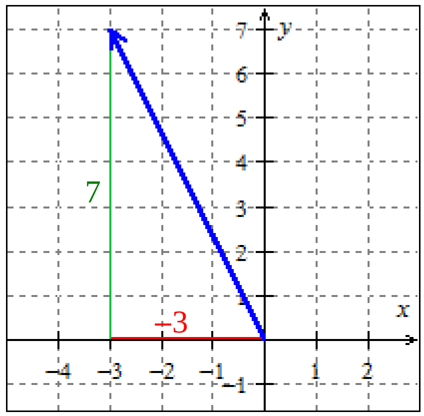

Figure 20: $\vec{p} = -3\vec{i} + 7\vec{j}$

We can use the Pythagorean Theorem to find $\|\vec{p}\|$, the magnitude of $\vec{p}$:

$$\begin{aligned} \|\vec{p}\|^2 &= (-3)^2 + (7)^2 \\ \Rightarrow \|\vec{p}\|^2 &= 9 + 49 \\ \Rightarrow \|\vec{p}\| &= \sqrt{58} \end{aligned}$$

So the magnitude of $\vec{p}$ is $\sqrt{58}$ units.

To describe the direction $\vec{p}$, we can find the angle that the vector makes with the positive x-axis; we've labeled this angle θ in Figure 21. To do this, we can first find the "reference angle" (labeled α in Figure 21) and then subtract this angle from 180° to find θ.

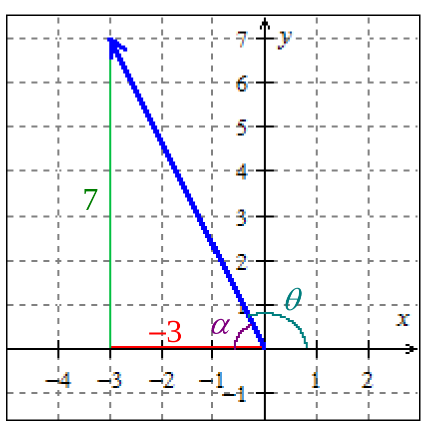

Figure 21

$$\tan(\alpha) = \frac{7}{3}$$ we use positive 3 since it represents a length in the "reference triangle".

$$\Rightarrow \quad \alpha = \tan^{-1}\left(\frac{7}{3}\right)$$

$$\Rightarrow \quad \alpha \approx 66.8^\circ$$

Thus,

$$\theta \approx 180^\circ - 66.8^\circ$$
$$= 113.2^\circ$$

so $\vec{p}$ makes an angle of about 113.2° with the positive x-axis.

EXAMPLE 7: Suppose that the vector $\vec{u}$ is represented by an arrow on the coordinate plane whose tail is at the point $(-2, 1)$ and tip is at the point $(5, 6)$. Find the components of $\vec{u}$.

SOLUTION:

First, lets draw the arrow that represents $\vec{u}$.

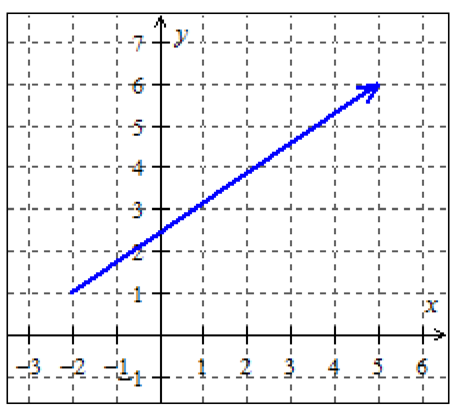

Figure 22: $\vec{u}$

To find the horizontal component of $\vec{u}$, we need to determine the difference in x-values between the tip and the tail:

$$5 - (-2) = 7$$

So the horizontal component is 7.

To find the vertical component of $\vec{u}$, we need to determine the difference in y-values between the tip and the tail:

$$6 - 1 = 5$$

So the vertical component is 5.

Thus, $\vec{u} = \langle 7, 5 \rangle = 7\vec{i} + 5\vec{j}$.

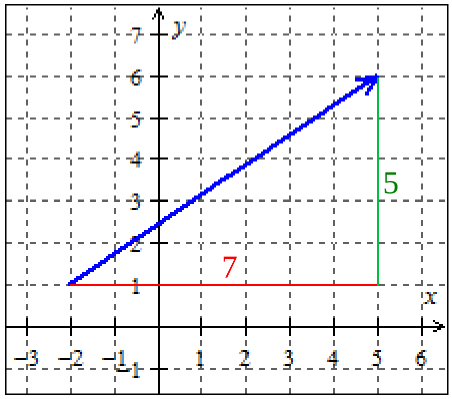

Figure 23: Thee components of $\vec{u}$.

> In general, if an arrow representing vector $\vec{v}$ has its tail at the point (x_1, y_1) and its tip at the point (x_2, y_2), then
>
> $$\begin{aligned} \vec{v} &= \langle (x_2 - x_1), (y_2 - y_1) \rangle \\ &= (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j}. \end{aligned}$$

EXAMPLE 8: Suppose that the vector $\vec{m}$ makes an angle of 37° with respect to the positive x-axis and that $\|\vec{m}\| = 20$. Find the horizontal and vertical components of $\vec{m}$.

SOLUTION:

First, lets draw the arrow that represents $\vec{m}$.

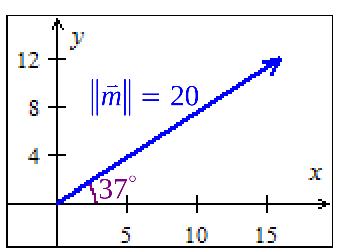

Figure 24: $\vec{m}$.

If we think of the arrow as being the hypotenuse of a right-triangle, we can use we can use right-triangle trigonometry to find the components of $\vec{m}$. (In Figure 25 we've labeled the horizontal component m_1 and the vertical component m_2.)

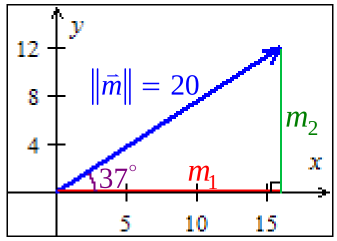

Figure 25

$$\cos(37^\circ) = \frac{m_1}{20} \qquad \text{and} \qquad \sin(37^\circ) = \frac{m_2}{20}$$

$$\Rightarrow \quad m_1 = 20\cos(37^\circ) \qquad\qquad \Rightarrow \quad m_2 = 20\sin(37^\circ)$$

$$\Rightarrow \quad m_1 \approx 15.97 \qquad\qquad \Rightarrow \quad m_2 \approx 12.04$$

Thus,

$$\begin{aligned}\vec{m} &= \left\langle 20\cos(37^\circ),\, 20\sin(37^\circ)\right\rangle \\ &\approx \left\langle 15.97, 12.04\right\rangle \\ &\approx 15.97\vec{i} + 12.04\vec{j}.\end{aligned}$$

In general, if vector $\vec{v}$ makes an angle θ with the positive x-axis then, in component form,

$$\begin{aligned}\vec{v} &= \left\langle \|\vec{v}\|\cos(\theta),\, \|\vec{v}\|\sin(\theta)\right\rangle \\ &= \|\vec{v}\|\cos(\theta)\,\vec{i} + \|\vec{v}\|\sin(\theta)\,\vec{j}.\end{aligned}$$

On the next page, we'll list some properties of vector addition and scalar multiplication.

Properties of Vector Addition and Scalar Multiplication

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors and a and b are scalars (i.e., $a, b \in \mathbb{R}$) then the following properties hold true:

1. **Commutativity of Vector Addition:** $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

2. **Associativity of Vector Addition:** $\left(\vec{u} + \vec{v}\right) + \vec{w} = \vec{u} + \left(\vec{v} + \vec{w}\right)$

3. **Associativity of Scalar Multiplication:** $a\left(b\vec{v}\right) = \left(ab\right)\vec{v}$

4. **Distributivity:** $\left(a + b\right)\vec{v} = a\vec{v} + b\vec{v}$

   and

   $a\left(\vec{u} + \vec{v}\right) = a\vec{u} + a\vec{v}$

5. **Identities:** $\vec{v} + \vec{0} = \vec{v}$ and $1 \cdot \vec{v} = \vec{v}$

You can check these properties by choosing some particular vectors and scalars and calculating the left and right side of each equation to see that they are equal.