Section II: Trigonometric Identities

Chapter 5: Double-Angle and Half-Angle Identities

In this chapter we will find identities that will allow us to calculate $\sin(2\theta)$ and $\cos(2\theta)$ if we know the values of $\cos(\theta)$ and $\sin(\theta)$ (we call these “double-angle identities”) and we will find identities that will allow us to calculate $\sin\left(\frac{\theta}{2}\right)$ and $\cos\left(\frac{\theta}{2}\right)$ if we know the values of $\cos(\theta)$ and $\sin(\theta)$ (we call these “half-angle identities”).

Let’s start by finding the **double-angle identities**. [Take note of how we will derive these identities differently than in our textbook. Our textbook uses the sum and difference identities but we'll use the laws of sine and cosine.] First we'll find an identity for $\sin(2\theta)$:

Recall that the definition of the cosine and sine functions tell us that $\cos(\theta)$ and $\sin(\theta)$ represent the horizontal and vertical coordinates, respectively, of the point specified by the angle θ on the unit circle; see Figure 1.

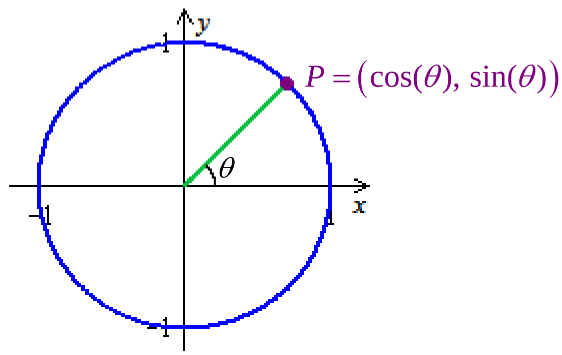

Figure 1: The unit circle with a point P specified by the angle θ.

We can construct a right triangle using the terminal side of angle θ. This triangle has hypotenuse of length 1 unit and sides of length $\cos(\theta)$ and $\sin(\theta)$; see Figure 2.

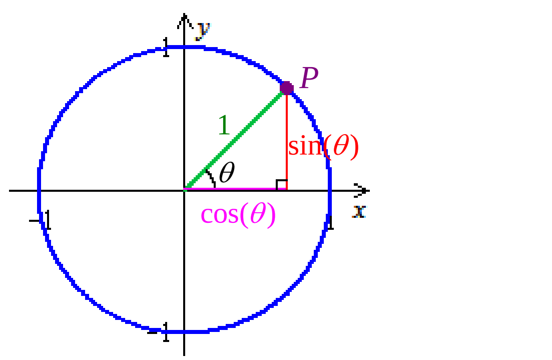

Figure 2: Right triangle with angle θ.

Although we will continue to focus on angle θ, let's label the angle whose vertex is point P; we'll call it α; see Figure 3.

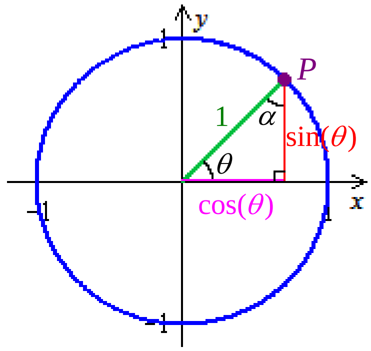

Figure 3

Notice that

$$\sin(\alpha) = \frac{\cos(\theta)}{1}$$
$$= \cos(\theta).$$

We'll use this fact later.

Now let's construct the mirror-image of this triangle below the x-axis in Quadrant IV; see Figure 4.

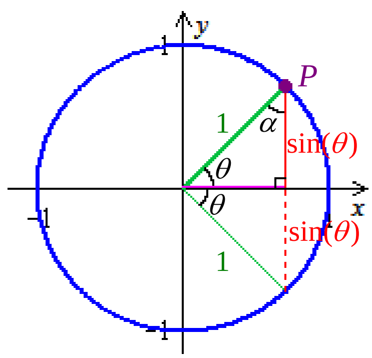

Figure 4

These two right triangles together form a larger non-right triangle that has an angle of measure 2θ; we've emphasized this triangle in Figure 5 by hiding the unit circle and the coordinate plane. (Note that the side opposite 2θ is of length $2\sin(\theta)$ since it consists of two segments each of length $\sin(\theta)$.)

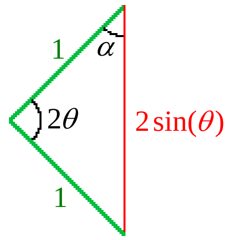

Figure 5

We can use this triangle to find the double-angle identities for cosine and sine. First, let's apply the **Law of Sines** to the triangle in Figure 5 to obtain the double-angle identity for sine.

The Law of Sines tells us that $\dfrac{\sin(2\theta)}{2\sin(\theta)} = \dfrac{\sin(\alpha)}{1}$; since $\sin(\alpha) = \cos(\theta)$ (from above), we can substitute $\cos(\theta)$ for $\sin(\alpha)$:

$$\frac{\sin(2\theta)}{2\sin(\theta)} = \frac{\sin(\alpha)}{1}$$

$$\Rightarrow \quad \frac{\sin(2\theta)}{2\sin(\theta)} = \frac{\cos(\theta)}{1}$$

$$\Rightarrow \quad \sin(2\theta) = 2\sin(\theta)\cos(\theta).$$

This last equation is the **double-angle identity for sine**. Notice that we can use this identity to obtain the value of $\sin(2\theta)$ if we know the values of $\cos(\theta)$ and $\sin(\theta)$.

Now let's find the double-angle identity for cosine. We can use the same triangle we constructed above (we've copied this triangle below in Figure 6), but apply the **Law of Cosines** instead of the Law of Sines.

$$\left(2\sin(\theta)\right)^2 = 1^2 + 1^2 - 2\cdot 1\cdot 1\cdot \cos(2\theta)$$

$$\Rightarrow \quad 4\sin^2(\theta) = 1 + 1 - 2\cos(2\theta)$$

$$\Rightarrow \quad 4\sin^2(\theta) = 2 - 2\cos(2\theta)$$

$$\Rightarrow \quad 2\cos(2\theta) = 2 - 4\sin^2(\theta)$$

$$\Rightarrow \quad \cos(2\theta) = 1 - 2\sin^2(\theta)$$

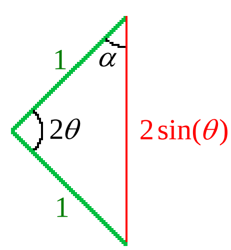

Figure 6

The last equation (above) is the **double-angle identity for cosine**. Notice that we can use this identity to obtain the value of $\cos(2\theta)$ if we know the value of $\sin(\theta)$. We can use the Pythagorean identity to obtain two other forms of the double-angle identity for cosine. Recall that the Pythagorean identity tells us that

$$\begin{aligned} & \sin^2(\theta) + \cos^2(\theta) = 1 \\ \Rightarrow \quad & \sin^2(\theta) = 1 - \cos^2(\theta), \end{aligned}$$

and we can now substitute $1 - \cos^2(\theta)$ for $\sin^2(\theta)$ in the double-angle identity to obtain another form of the identity:

$$\begin{aligned} & \cos(2\theta) = 1 - 2\sin^2(\theta) \\ \Rightarrow \quad & \cos(2\theta) = 1 - 2\left(1 - \cos^2(\theta)\right) \\ \Rightarrow \quad & \cos(2\theta) = 1 - 2 + 2\cos^2(\theta) \\ \Rightarrow \quad & \cos(2\theta) = 2\cos^2(\theta) - 1. \end{aligned}$$

This last equation is another double-angle identity for cosine. We can obtain a third double-angle identity for cosine by substituting $\sin^2(\theta) + \cos^2(\theta)$ for 1:

$$\begin{aligned} & \cos(2\theta) = 2\cos^2(\theta) - 1 \\ \Rightarrow \quad & \cos(2\theta) = 2\cos^2(\theta) - \left(\sin^2(\theta) + \cos^2(\theta)\right) \\ \Rightarrow \quad & \cos(2\theta) = 2\cos^2(\theta) - \sin^2(\theta) - \cos^2(\theta) \\ \Rightarrow \quad & \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta). \end{aligned}$$

Below, we've summarized the double-angle identities for sine and cosine:

DOUBLE-ANGLE IDENTITIES

sine : $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

cosine : $\begin{cases} \cos(2\theta) = 1 - 2\sin^2(\theta) \\ \cos(2\theta) = 2\cos^2(\theta) - 1 \\ \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \end{cases}$

We could easily find double-angle identities for tangent by using the fact that $\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)}$. But the resulting identities aren't easy to remember so it makes more sense to learn the identities for sine and cosine and use the fact that $\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)}$ if you ever need to calculate $\tan(2\theta)$.

EXAMPLE 1: Suppose that $\sin(\alpha) = \frac{1}{3}$ and that α is in Quadrant II. Find $\sin(2\alpha)$, $\cos(2\alpha)$, and $\tan(2\alpha)$.

SOLUTION:

First, let's find $\cos(\alpha)$ since we need this value to use the double-angle identity for sine. To find $\cos(\alpha)$, let's use the Pythagorean identity:

$$\begin{aligned}
& \sin^2(\alpha) + \cos^2(\alpha) = 1 \\
\Rightarrow \quad & \left(\tfrac{1}{3}\right)^2 + \cos^2(\alpha) = 1 \\
\Rightarrow \quad & \tfrac{1}{9} + \cos^2(\alpha) = 1 \\
\Rightarrow \quad & \cos^2(\alpha) = 1 - \tfrac{1}{9} \\
\Rightarrow \quad & \cos^2(\alpha) = \tfrac{8}{9} \\
\Rightarrow \quad & \cos(\alpha) = -\tfrac{2\sqrt{2}}{3}.
\end{aligned}$$

Note that we take the negative square root of $\frac{8}{9}$ since α is in Quadrant II so that $\cos(\alpha)$ must be negative.

Now we can use the double-angle identities to find $\sin(2\alpha)$ and $\cos(2\alpha)$. Let's start with $\sin(2\alpha)$:

$$\begin{aligned}
\sin(2\alpha) &= 2\sin(\alpha)\cos(\alpha) \\
&= 2\left(\tfrac{1}{3}\right)\left(-\tfrac{2\sqrt{2}}{3}\right) \\
&= -\tfrac{4\sqrt{2}}{9}.
\end{aligned}$$

To find $\cos(2\alpha)$, we can use any one of the three double-angle identities for cosine. Let's use $\cos(2\alpha) = 1 - 2\sin^2(\alpha)$:

$$\begin{aligned}\cos(2\alpha) &= 1 - 2\sin^2(\alpha)\\ &= 1 - 2\left(\tfrac{1}{3}\right)^2\\ &= 1 - 2\cdot\tfrac{1}{9}\\ &= 1 - \tfrac{2}{9}\\ &= \tfrac{7}{9}.\end{aligned}$$

You should verify that the other double-angle identities for cosine give the same value for $\cos(2\alpha)$.

To find $\tan(2\alpha)$, we can use the fact that $\tan(2\alpha) = \dfrac{\sin(2\alpha)}{\cos(2\alpha)}$:

$$\begin{aligned}\tan(2\alpha) &= \frac{\sin(2\alpha)}{\cos(2\alpha)}\\ &= \frac{-\frac{4\sqrt{2}}{9}}{\frac{7}{9}}\\ &= -\frac{4\sqrt{2}}{9}\cdot\frac{9}{7}\\ &= -\frac{4\sqrt{2}}{7}.\end{aligned}$$

Half-Angle Identities

We can use the double-angle identities for cosine to derive **half-angle identities**.

Recall that $\cos(2\theta) = 1 - 2\sin^2(\theta)$ we can use this identity to find a half-angle identity for sine. Let $\alpha = 2\theta$. Then $\theta = \frac{\alpha}{2}$ and

$$\begin{aligned}
&\cos(2\theta) = 1 - 2\sin^2(\theta) \\
\Rightarrow\quad &\cos(\alpha) = 1 - 2\sin^2\left(\tfrac{\alpha}{2}\right) \\
\Rightarrow\quad &2\sin^2\left(\tfrac{\alpha}{2}\right) = 1 - \cos(\alpha) \\
\Rightarrow\quad &\sin^2\left(\tfrac{\alpha}{2}\right) = \frac{1 - \cos(\alpha)}{2} \\
\Rightarrow\quad &\sin\left(\tfrac{\alpha}{2}\right) = \pm\sqrt{\frac{1 - \cos(\alpha)}{2}}
\end{aligned}$$

(we choose the sign based on the quadrant of $\frac{\alpha}{2}$)

We can use $\cos(2\theta) = 2\cos^2(\theta) - 1$ to find a half-angle identity for cosine. Again, suppose that $\alpha = 2\theta$. Then $\theta = \frac{\alpha}{2}$ and

$$\begin{aligned}
&\cos(2\theta) = 2\cos^2(\theta) - 1 \\
\Rightarrow\quad &\cos(\alpha) = 2\cos^2\left(\tfrac{\alpha}{2}\right) - 1 \\
\Rightarrow\quad &1 + \cos(\alpha) = 2\cos^2\left(\tfrac{\alpha}{2}\right) \\
\Rightarrow\quad &\frac{1 + \cos(\alpha)}{2} = \cos^2\left(\tfrac{\alpha}{2}\right) \\
\Rightarrow\quad &\cos\left(\tfrac{\alpha}{2}\right) = \pm\sqrt{\frac{1 + \cos(\alpha)}{2}}
\end{aligned}$$

(we choose the sign based on the quadrant of $\frac{\alpha}{2}$)

Since these identities are valid for *any* value of α, we can express the identities in terms of θ since that's how we've been expressing our identities.

HALF-ANGLE IDENTITIES

sine : $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos(\theta)}{2}}$ (we choose the sign based on the quadrant of $\frac{\theta}{2}$)

cosine : $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 + \cos(\theta)}{2}}$ (we choose the sign based on the quadrant of $\frac{\theta}{2}$)

EXAMPLE 2: Recall from Example 1 that $\sin(\alpha) = \frac{1}{3}$ and that α is in Quadrant II. Find $\cos\left(\frac{\alpha}{2}\right)$, $\sin\left(\frac{\alpha}{2}\right)$, and $\tan\left(\frac{\alpha}{2}\right)$.

SOLUTION:

First, note that since α is in Quadrant II, $\frac{\pi}{2} < \alpha < \pi$. Thus,

$$\frac{\pi/2}{2} < \frac{\alpha}{2} < \frac{\pi}{2}$$
$$\Rightarrow \frac{\pi}{4} < \frac{\alpha}{2} < \frac{\pi}{2}$$

Thus, $\frac{\alpha}{2}$ is in Quadrant I so it's sine and cosine values are positive.

In Example 1 we used the Pythagorean identity to determine that $\cos(\alpha) = -\frac{2\sqrt{2}}{3}$; we can use this value and the half-angle identities to find $\cos\left(\frac{\alpha}{2}\right)$ and $\sin\left(\frac{\alpha}{2}\right)$:

$$\begin{aligned}
\sin\left(\frac{\alpha}{2}\right) &= +\sqrt{\frac{1-\cos(\alpha)}{2}} \qquad \text{(we choose + since } \tfrac{\alpha}{2} \text{ is in Quadrant I)} \\
&= \sqrt{\frac{\left(\frac{3}{3} - \left(-\frac{2\sqrt{2}}{3}\right)\right)}{2}} \\
&= \sqrt{\frac{1}{2}\left(\frac{3+2\sqrt{2}}{3}\right)} \\
&= \sqrt{\frac{3+2\sqrt{2}}{6}}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\cos\left(\frac{\alpha}{2}\right) &= +\sqrt{\frac{1+\cos(\alpha)}{2}} \qquad \text{(we choose + since } \tfrac{\alpha}{2} \text{ is in Quadrant I)} \\
&= \sqrt{\frac{\left(\frac{3}{3} + \left(-\frac{2\sqrt{2}}{3}\right)\right)}{2}} \\
&= \sqrt{\frac{1}{2}\left(\frac{3-2\sqrt{2}}{3}\right)} \\
&= \sqrt{\frac{3-2\sqrt{2}}{6}}
\end{aligned}$$

Now we can find $\tan\left(\frac{\alpha}{2}\right)$ using the fact that $\tan\left(\frac{\alpha}{2}\right) = \frac{\sin\left(\frac{\alpha}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)}$:

$$\begin{aligned}\tan\left(\frac{\alpha}{2}\right) &= \frac{\sin\left(\frac{\alpha}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)}\\ &= \frac{\sqrt{\frac{3+2\sqrt{2}}{6}}}{\sqrt{\frac{3-2\sqrt{2}}{6}}}\\ &= \frac{\sqrt{3+2\sqrt{2}}}{\sqrt{3-2\sqrt{2}}}\end{aligned}$$

Note that it's easy to derive a half-angle identity for tangent but, as we discussed when we studied the double-angle identities, we can always use sine and cosine values to find tangent values so there's no reason to waste energy on additional identities for tangent.