Section II: Trigonometric Identities

Chapter 2: The Laws of Sines and Cosines

In Section I, Chapter 9, we studied right triangle trigonometry and learned how we can use the sine and cosine functions to obtain information about right triangles. In this section we'll study how we can use sine and cosine to obtain information about non-right triangles. The triangle in Figure 1 is a non-right triangle since none of its angles measure 90°.

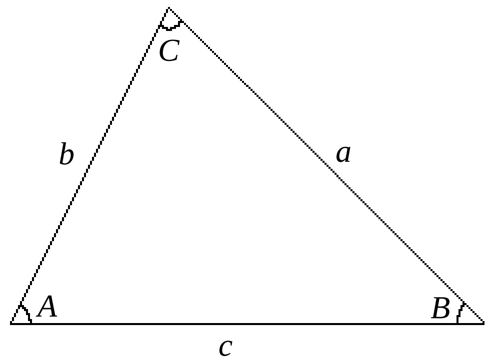

Figure 1

Let's derive the Laws of Sines and Cosines so that we can study non-right triangles. The laws are **identities** since they are true for *all* triangles.

The Law of Sines

To derive the Law of Sines, let's construct a segment h in the triangle given in Figure 1 that connects the vertex of angle C to the side c; this segment should be perpendicular to side c and is called a *height* of the triangle; see Figure 2:

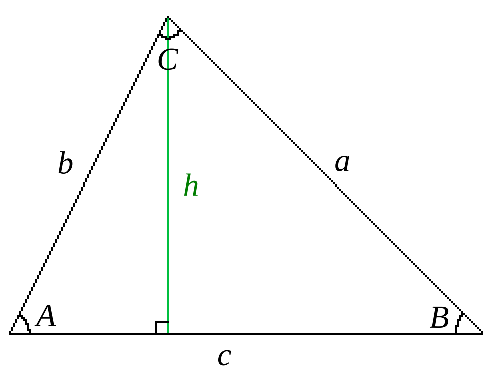

Figure 2

The segment h splits the triangle into two right triangles on which we can apply what we know about right triangle trigonometry; see Figure 3.

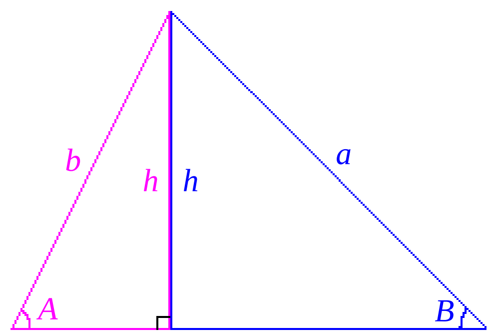

Figure 3

We can use the two right triangles in Figure 3 to obtain expressions for both $\sin(A)$ and $\sin(B)$:

$$\sin(A) = \frac{h}{b} \quad \text{and} \quad \sin(B) = \frac{h}{a}$$

We can now solve both of these equations for h:

$$\sin(A) = \frac{h}{b} \quad \text{and} \quad \sin(B) = \frac{h}{a}$$

$$\Rightarrow \; h = b\sin(A) \quad \text{and} \quad h = a\sin(B)$$

Now, since both of the h's represent the length of the same segment, they are equal. By setting the h's equal to each other we obtain the following:

$$b\sin(A) = a\sin(B)$$

This equation provides us with what is known as the Law of Sines. Typically, the law is written in terms of ratios. If we divide both sides by $a \cdot b$ we obtain the following.

$$b\sin(A) = a\sin(B)$$

$$\Rightarrow \frac{b\sin(A)}{a \cdot b} = \frac{a\sin(B)}{a \cdot b}$$

$$\Rightarrow \quad \frac{\sin(A)}{a} = \frac{\sin(B)}{b}$$

(Note that the law also holds with angle C and side c since the analysis we've shown above also holds if we focus on this angle and side.)

THE LAW OF SINES

If a triangle's sides and angles are labeled like the triangle in Figure 4 then

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$$

This is an **identity** since it is true for *all* triangles.

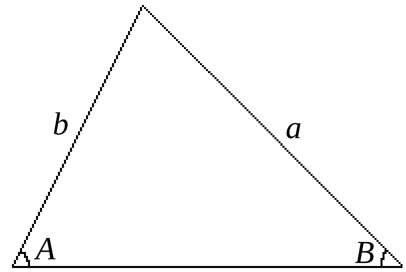

Figure 4

EXAMPLE 1: Find all of the missing angles and side-lengths of the triangle given in Figure 5. (The triangle is not necessarily drawn to scale.)

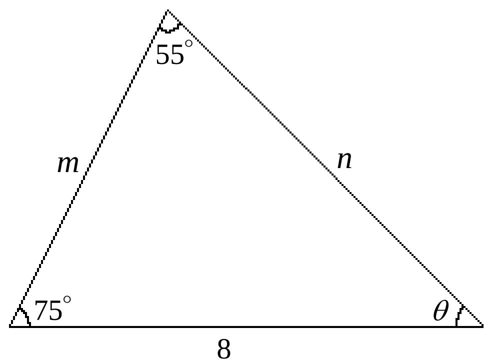

Figure 5

SOLUTION:

We can easily find θ since we know that the sum of the angle-measures in a triangle is always 180°. So

$$\theta + 55^\circ + 75^\circ = 180^\circ$$

$$\Rightarrow \qquad \theta = 50^\circ.$$

Now we can use the Law of Sines to find m and n. (Be sure your calculator is in degree mode to approximate these values.)

$$\frac{\sin(\theta)}{m} = \frac{\sin(55^\circ)}{8} \quad \Rightarrow \quad \frac{m}{\sin(\theta)} = \frac{8}{\sin(55^\circ)}$$

$$\Rightarrow \frac{m}{\sin(50^\circ)} = \frac{8}{\sin(55^\circ)}$$

$$\Rightarrow \qquad m = \frac{8 \cdot \sin(50^\circ)}{\sin(55^\circ)} \approx 7.48$$

and

$$\frac{\sin(75^\circ)}{n} = \frac{\sin(55^\circ)}{8} \quad \Rightarrow \quad \frac{n}{\sin(75^\circ)} = \frac{8}{\sin(55^\circ)}$$

$$\Rightarrow \qquad n = \frac{8\cdot\sin(75^\circ)}{\sin(55^\circ)} \approx 9.43$$

In Figure 6 we've drawn the given triangle and included all of its angle and side-length measures.

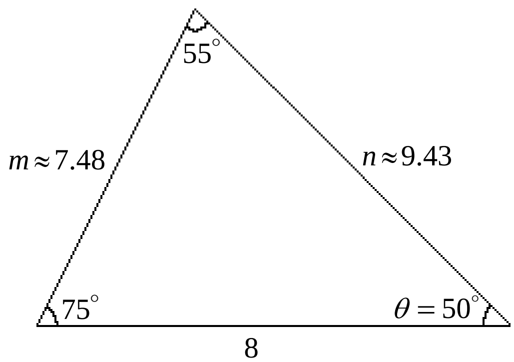

Figure 6

Notice that the longest side is opposite the largest angle and the shortest side is opposite the smallest angle. This is *always* true for a triangle and a *very* helpful fact to keep in mind so that you can make sure that your answer is even possibly correct.

Notice that the Law of Sines involves two angles and the two sides opposite those angles. In order to use the Law of Sines to find a missing part of missing part of a triangle, we need to know three of these for things. So the Law of Sines only is helpful if we know the length of two of the sides and the measure of the angle opposite one of these sides or if we know the measure of two angles and the length of the side opposite one of these angles. In the example above, we were able to use the Law of Sines since we were given the measure of two angles and the length of the side opposite one of these angles. Consider the triangle in Fig. 7:

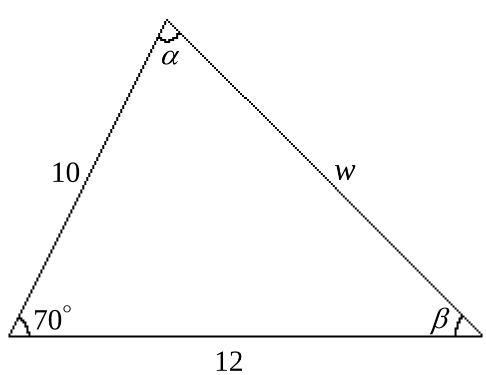

Figure 7

In this triangle, we are not given enough information to use the Law of Sines since we aren't given any "angle and side opposite" combination. (In other words, if we know a side's length, we don't know the opposite angle's measure, and if we know the angle's measure, we don't know the opposite side's length.) In order to find the missing measurements of this triangle, we need another law: the **Law of Cosines**. Let's derive the Law of Cosines just as we derived the Law of Sines earlier in this chapter.

The Law of Cosines

Let's start with a generic triangle and draw the height, h, just as we did at the previously in this chapter; see Figure 8.

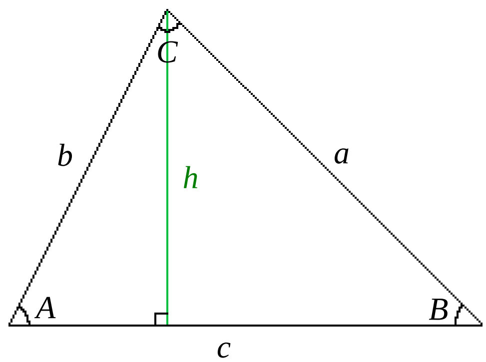

Figure 8

Again we want to consider the two right triangles induced by constructing the line h on the triangle given in Figure 8. This time we want to use the two pieces that the side c is split into. Let's call the segment on the left (the one closest to angle A) x and then the segments on the right must be $c-x$ units long. We've emphasized the two right triangles and labeled the two pieces of side c in Figure 9.

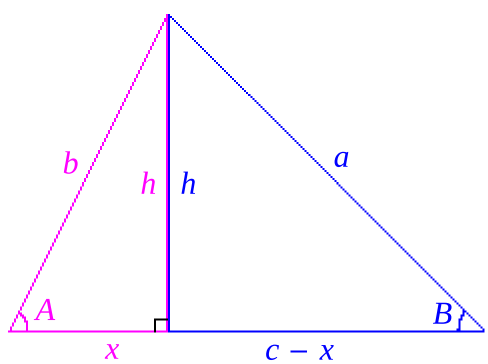

Figure 9

First, notice that the following is true:

$$\cos(A) = \tfrac{x}{b}$$

$$\Rightarrow \qquad x = b\cos(A).$$

We'll use this fact later.

Now let's apply the Pythagorean Theorem to each of the two triangles in Figure 9. The pink triangle on the left gives us

$$x^2 + h^2 = b^2,$$

and we can solve this for h^2 and obtain

$$h^2 = b^2 - x^2.$$

The blue triangle on the right gives us

$$(c - x)^2 + h^2 = a^2$$

and we can use the fact that $h^2 = b^2 - x^2$ to eliminate h from this equation:

$$\begin{aligned} &(c - x)^2 + h^2 = a^2 \\ \Rightarrow\ &(c - x)^2 + (b^2 - x^2) = a^2. \end{aligned}$$

Finally, we can simplify the left side of this equation and use the fact that $x = b\cos(A)$ to eliminate x:

$$\begin{aligned} &(c - x)^2 + (b^2 - x^2) = a^2 \\ \Rightarrow\ &c^2 - 2cx + x^2 + b^2 - x^2 = a^2 \\ \Rightarrow\ &c^2 - 2cx + b^2 = a^2 \\ \Rightarrow\ &c^2 - 2c \cdot b\cos(A) + b^2 = a^2 \qquad (\text{since } x = b\cos(A)) \end{aligned}$$

This last equation is known as the Law of Cosines. Below we've re-written the law in its standard form.

THE LAW OF COSINES

If a triangle's sides and angles are labeled like the triangle in Figure 10 then

$$a^2 = b^2 + c^2 - 2bc \cdot \cos(A)$$

This is an **identity** since it is true for *all* triangles.

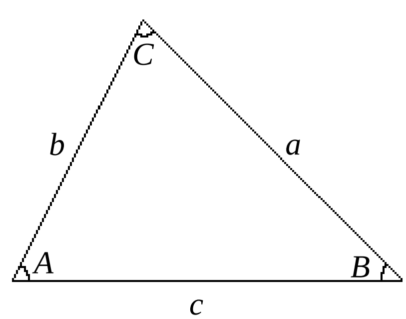

Figure 10

Notice that when $A = 90^\circ$, the Law of Cosines is equivalent to the Pythagorean theorem. (Verify this by substituting 90° for A.) For this reason, the **Law of Cosines is considered the generalization of the Pythagorean Theorem**.

EXAMPLE 2: Find all of the missing angles and side-lengths of the triangle given in Figure 11. (The triangle is not necessarily drawn to scale.)

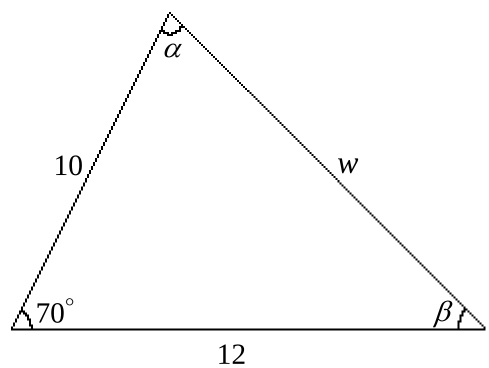

Figure 11

SOLUTION:

We can use the Law of Cosines to find w. (Be sure your calculator is in degree mode to approximate this value.)

$$
\begin{aligned}
& w^2 = 12^2 + 10^2 - 2(12)(10)\cdot\cos(70^\circ) \\
\Rightarrow\ & w^2 = 144 + 100 - 240\cdot\cos(70^\circ) \\
\Rightarrow\ & w = \sqrt{144 + 100 - 240\cdot\cos(70^\circ)} \\
\Rightarrow\ & w \approx 12.725
\end{aligned}
$$

Now that we know w, we now know an angle and the side opposite that angle, so we can use the Law of Sines to find either of the other angles. Let's find β.

$$
\begin{aligned}
& \frac{\sin(\beta)}{10} = \frac{\sin(70^\circ)}{w} \\
\Rightarrow\ & \sin(\beta) = \frac{10\cdot\sin(70^\circ)}{w} \\
\Rightarrow\ & \sin^{-1}\left(\sin(\beta)\right) = \sin^{-1}\left(\frac{10\cdot\sin(70^\circ)}{w}\right) \\
\Rightarrow\ & \beta = \sin^{-1}\left(\frac{10\cdot\sin(70^\circ)}{w}\right) \\
\Rightarrow\ & \beta \approx 47.6^\circ \quad \text{using the fact that } w \approx 12.725
\end{aligned}
$$

Finally, we can find α by using the fact that the sum of the angle-measures in a triangle is always 180°:

$$\begin{aligned} \alpha + \beta + 70^\circ &= 180^\circ \\ \Rightarrow \quad \alpha + 47.6^\circ + 70^\circ &\approx 180^\circ \\ \Rightarrow \quad \alpha &\approx 62.4^\circ \end{aligned}$$

In Figure 12 we've drawn the given triangle and included all of its angle and side-length measures.

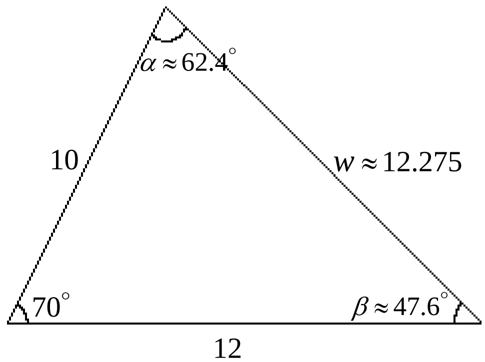

Figure 12

Notice that as with the triangle given in Figure 6 (and with *all* triangles) the longest side is opposite the longest angle and the smallest side is opposite the smallest angle. If this weren't the case, we would know that our answer was incorrect and know that we've made a mistake.

EXAMPLE 3: If a triangle has a side of length 6 units, a side of length 8 units, and the angle opposite the side of length 6 units measures 35°, find the missing side-length and the missing angle-measures.

SOLUTION:

First, we need to translate the information on a drawn triangle; see Figure 13.

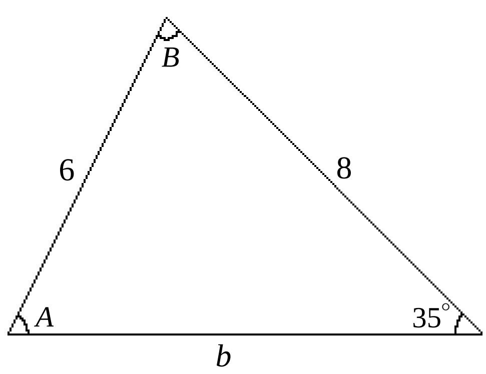

Figure 13

The way we've drawn the triangle could reasonable, but the information we've been given is ambiguous enough that there is an entirely different way we could draw a triangle with the given angle measure and side lengths. Imagine pivoting the 6 unit segment at the vertex of angle B. Since this segment is shorter than the 8 unit segment, it's possible that it intersects segment b at another location closer to the 35° angle. This will create a second possible angle A and a corresponding second possible angle B and second possible side b. In Figure 14, we've shown the two possible triangles with the same given information.

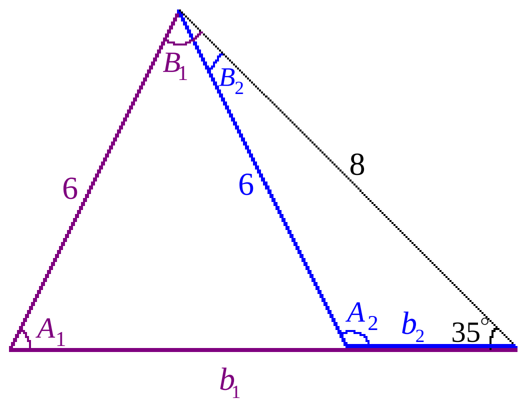

Figure 14

We'll start with the triangle containing A_1, B_1, and b_1. (Part of this triangle is purple in Figure 14.) First, let's use the Law of Sines to find A_1:

$$\frac{\sin(A_1)}{8} = \frac{\sin(35^\circ)}{6}$$

$$\Rightarrow \qquad \sin(A_1) = \frac{8\sin(35^\circ)}{6}$$

$$\Rightarrow \sin^{-1}\left(\sin(A_1)\right) = \sin^{-1}\left(\frac{8\sin(35^\circ)}{6}\right)$$

$$\Rightarrow \qquad A_1 = \sin^{-1}\left(\frac{8\sin(35^\circ)}{6}\right)$$

$$\Rightarrow \qquad A_1 \approx 49.9^\circ.$$

We can now find B_1 by using the fact that the sum of the angle-measures in a triangle is always 180°:

$$A_1 + B_1 + 35^\circ = 180^\circ$$

$$\Rightarrow 49.9^\circ + B_1 + 35^\circ \approx 180^\circ$$

$$\Rightarrow \qquad B_1 \approx 95.1^\circ.$$

Now that we know the measure of the angle opposite b_1, we can use the Law of Sines to find it:

$$\frac{\sin(B_1)}{b_1} = \frac{\sin(35^\circ)}{6} \quad \Rightarrow \quad \frac{\sin(95.1^\circ)}{b_1} \approx \frac{\sin(35^\circ)}{6}$$

$$\Rightarrow \quad \frac{b_1}{\sin(95.1^\circ)} \approx \frac{6}{\sin(35^\circ)}$$

$$\Rightarrow \quad b_1 \approx \frac{6 \cdot \sin(95.1^\circ)}{\sin(35^\circ)} \approx 10.42.$$

These values for A_1, B_1, and b_1 give us possible triangle with the given measurements. We've drawn this triangle (not to scale) in Figure 15.

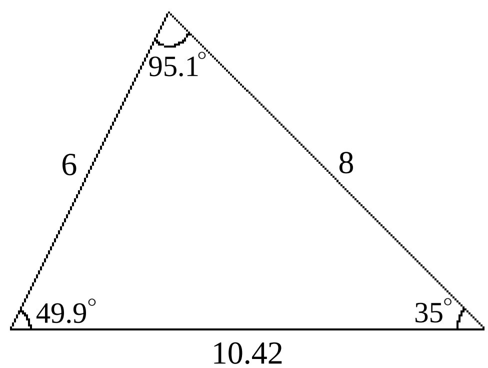

Figure 15

We've done everything correctly and found **one** triangle using all of the given information…so how will we find the values of A_2, B_2, and b_2 for the other triangle we drew in Figure 14? In order to find this other possible triangle, we have to remember the problem with using the inverse sine function, which we used to find A_1. Recall that the inverse sine function has range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; thus, when we use the inverse sine function we will *only* obtain acute angles (i.e., angles that measure less than 90°). In order to find the obtuse (i.e., greater than 90°) possibility, we need to use the identity $\sin(\theta) = \sin(\pi - \theta)$ that we first noticed in Section I: Chapter 3. (Since we're currently working with degrees instead of radians, let's use $\sin(\theta) = \sin(180^\circ - \theta)$ instead.) Above we found that

$$\sin(A_1) = \frac{8\sin(35^\circ)}{6}$$

$$\Rightarrow \quad A_1 = \sin^{-1}\left(\frac{8\sin(35^\circ)}{6}\right) \approx 49.9^\circ.$$

The identity tells us that $\sin(A_1) = \sin(180^\circ - A_1)$, so we can let $A_2 = 180^\circ - A_1$ and we will

have another angle, A_2, such that $\sin(A_2) = \dfrac{8\sin(35^\circ)}{6}$. Thus,

$$\begin{aligned} A_2 &= 180^\circ - A_1 \\ &\approx 180^\circ - 49.9^\circ \approx 130.1^\circ. \end{aligned}$$

We can now find B_2 by using the fact that the sum of the angle-measures in a triangle is always 180°:

$$\begin{aligned} A_2 + B_2 + 35^\circ = 180^\circ \quad &\Rightarrow \quad 130.1^\circ + B_2 + 35^\circ \approx 180^\circ \\ &\Rightarrow \quad B_2 \approx 14.9^\circ. \end{aligned}$$

Finally, we can use the Law of Sines to find b_2:

$$\begin{aligned} \frac{\sin(B_2)}{b_2} = \frac{\sin(35^\circ)}{6} \quad &\Rightarrow \quad \frac{\sin(14.9^\circ)}{b_2} \approx \frac{\sin(35^\circ)}{6} \\ &\Rightarrow \quad \frac{b_2}{\sin(14.9^\circ)} \approx \frac{6}{\sin(35^\circ)} \\ &\Rightarrow \quad b_2 \approx \frac{6 \cdot \sin(14.9^\circ)}{\sin(35^\circ)} \approx 2.69. \end{aligned}$$

These values for A_2, B_2, and b_2 give us another possible triangle with the given measurements. We've drawn this triangle (not to scale) in Figure 16.

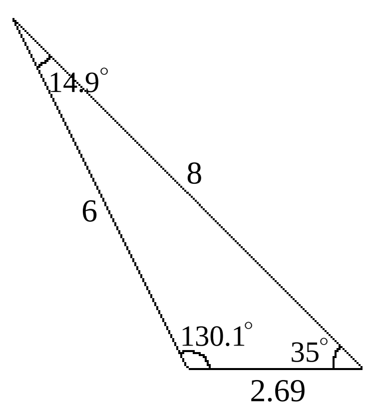

Figure 16

Whenever we're given two side-lengths and one angle measure and use the Law of Sines, it's possible that there will be two different triangles that satisfy the given information.