

Section III: Polar Coordinates and Complex Numbers

Chapter 3: Complex Numbers

In this chapter we'll study how we can employ what we know about polar coordinates and trigonometry to represent complex numbers. Let's start by reviewing complex numbers.

Recall from Section I: Chapter 0 the definition of the set of complex numbers:

$$\mathbb{C} = \left\{ x \mid x = a + bi \text{ and } a, b \in \mathbb{R} \text{ and } i = \sqrt{-1} \right\}.$$

If a complex number has the form $a + bi$, we say that its *real part* is a and its *imaginary part* is b .



EXAMPLE 1: Which of the following is a complex number?

- a.** $s = 2 + 5i$ **b.** $t = \frac{3}{2} - 3i$ **c.** $u = 3i$ **d.** $v = -4$

SOLUTION:

- a.** $s = 2 + 5i$ is a complex number since it has the form $a + bi$ where $a = 2$ and $b = 5$. The real part of s is 2 and the imaginary part of s is 5.
- b.** $t = \frac{3}{2} - 3i$ is a complex number since it has the form $a + bi$ where $a = \frac{3}{2}$ and $b = -3$. The real part of t is $\frac{3}{2}$ and the imaginary part of t is -3 .
- c.** $u = 3i$ is a complex number since it has the form $a + bi$ where $a = 0$ and $b = 3$. The real part of u is 0 and the imaginary part of u is 3.
- d.** $v = -4$ is a complex number since it has the form $a + bi$ where $a = -4$ and $b = 0$. The real part of v is -4 and the imaginary part of v is 0.

Because a complex number has *two* parts, we can use the *two dimensional* rectangular coordinate plane to plot complex numbers. We use the horizontal axis to represent the real part and the vertical axis to represent the complex part. Thus, the complex number $a + bi$ can be represented by the point (a, b) on the rectangular coordinate plane; see Figure 1.

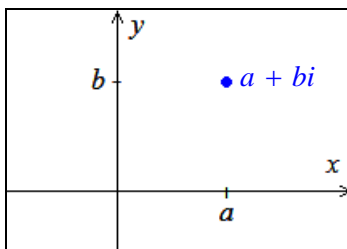


Figure 1



EXAMPLE 2: Plot the following complex numbers on the coordinate plane in Figure 2.

- a.** $s = 2 + 5i$ **b.** $t = \frac{3}{2} - 3i$ **c.** $u = 3i$ **d.** $v = -4$

SOLUTION:

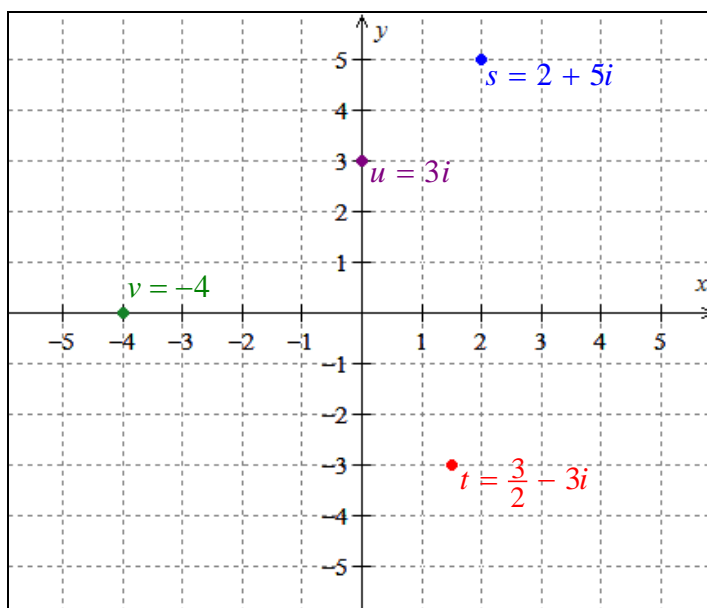


Figure 2

As we studied in Section III: Chapter 1, the rectangular ordered pair (a, b) can be represented in polar coordinates (r, θ) where r represents the distance the point is from the origin and θ represents the angle between the positive x -axis and the segment connecting the origin and the point; see Figure 3.

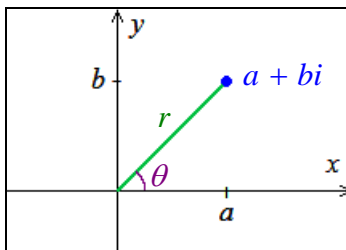


Figure 3

We know that if the rectangular pair (a, b) represents the same point as the polar pair (r, θ) , then $a = r \cos(\theta)$ and $b = r \sin(\theta)$. Thus,

$$\begin{aligned} a + bi &= r \cos(\theta) + r \sin(\theta) \cdot i \\ &= r(\cos(\theta) + \sin(\theta) \cdot i) \end{aligned}$$

Using this fact, we can establish a surprising connection between exponential function e^x and complex numbers. If you continue studying mathematics and take a calculus sequence, you have an opportunity to see why this equation is true but, for now, you need to just accept it and learn to work with it.

EULER'S FORMULA

$$e^{i\theta} = \cos(\theta) + \sin(\theta) \cdot i$$

By multiplying both sides of Euler's formula by r , we obtain the formula

$$re^{i\theta} = r \cos(\theta) + r \sin(\theta) \cdot i$$

that allows us to write any complex number in **polar form**:

The **polar form** of the complex number $z = a + bi$ is

$$z = re^{i\theta}$$

where $r = \sqrt{a^2 + b^2}$ and $\tan(\theta) = \frac{b}{a}$.



EXAMPLE: Express the complex number $z = 6e^{i \cdot \frac{5\pi}{6}}$ in the form $z = a + bi$.

SOLUTION:

$$\begin{aligned} z &= 6e^{i \cdot \frac{5\pi}{6}} \\ &= 6\cos\left(\frac{5\pi}{6}\right) + 6\sin\left(\frac{5\pi}{6}\right) \cdot i \\ &= 6 \cdot \left(-\frac{\sqrt{3}}{2}\right) + 6 \cdot \left(\frac{1}{2}\right) \cdot i \\ &= -3\sqrt{3} + 3i \end{aligned}$$

Thus, the complex number $z = 6e^{i \cdot \frac{5\pi}{6}}$ can be expressed as $z = -3\sqrt{3} + 3i$.



EXAMPLE: Find the polar form $z = re^{i\theta}$ of the complex number $z = 3 - 3i$.

SOLUTION:

We can associate the complex number $z = 3 - 3i$ with the rectangular ordered pair $(3, -3)$, and then translate this ordered pair into polar coordinates (r, θ) , and finally use this polar ordered pair to obtain the polar form $z = re^{i\theta}$. First, let's find r :

$$\begin{aligned} r &= \sqrt{(3)^2 + (-3)^2} \\ &= \sqrt{9 + 9} \\ &= 3\sqrt{2}. \end{aligned}$$

Now, let's find θ :

$$\begin{aligned} \tan(\theta) &= \frac{-3}{3} \\ \Rightarrow \theta &= \tan^{-1}(-1) \\ \Rightarrow \theta &= -\frac{\pi}{4} \end{aligned}$$

Thus, the complex number $z = 3 - 3i$ can be expressed in polar form $z = 3\sqrt{2} e^{i \cdot \left(-\frac{\pi}{4}\right)}$
