Section III: Polar Coordinates and Complex Numbers

Chapter 2: Polar Equations and Functions

Just as we can create equations in rectangular coordinates, we can create equations in polar coordinates. For example, $r = 3\sin(\theta)$ is an equation in polar coordinates since it's an equation and it involves the polar coordinates r and θ.

When we have an equation in one coordinate system, we can often convert it into an equation in another coordinate system. In Examples 1 and 2, we'll convert a polar equation into a rectangular equation, and vice versa.

EXAMPLE 1: Convert the polar equation $r = 3\sin(\theta)$ into an equivalent equation in rectangular coordinates.

SOLUTION:

As we know, the following identities can be used to convert from polar coordinates $(r,\ \theta)$ to rectangular coordinates $(x,\ y)$:

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

Since the equation $r = 3\sin(\theta)$ involves r, we'll certainly employ the identity $r = \sqrt{x^2 + y^2}$. Further, since the equation involves $\sin(\theta)$, we'll need to get an expression for $\sin(\theta)$ that contains only x and y (i.e., only rectangular coordinates):

$$\begin{aligned} & y = r\sin(\theta) \\ \Rightarrow \quad & \sin(\theta) = \frac{y}{r} \\ \Rightarrow \quad & \sin(\theta) = \frac{y}{\sqrt{x^2 + y^2}} \qquad \text{since } r = \sqrt{x^2 + y^2} \end{aligned}$$

Thus,

$$\begin{aligned} r &= 3\sin(\theta) \\ \Rightarrow \sqrt{x^2 + y^2} &= 3 \cdot \frac{y}{\sqrt{x^2 + y^2}} \\ \Rightarrow \quad x^2 + y^2 &= 3y \end{aligned}$$

So the equation $r = 3\sin(\theta)$ in polar coordinates is equivalent to the equation $x^2 + y^2 = 3y$ in rectangular coordinates.

EXAMPLE 2: Convert the rectangular equation $y = 4x - 3$ into an equivalent equation in polar coordinates.

SOLUTION:

As we know, the following identities can be used to convert from rectangular coordinates (x, y) to polar coordinates (r, θ):

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

Thus,

$$\begin{aligned} y &= 4x - 3 \\ \Rightarrow \qquad r\sin(\theta) &= 4 \cdot r\cos(\theta) - 3 \\ \Rightarrow \quad r\sin(\theta) - 4r\cos(\theta) &= -3 \\ \Rightarrow \quad r\left(\sin(\theta) - 4\cos(\theta)\right) &= -3 \\ \Rightarrow \qquad r &= -\frac{3}{\sin(\theta) - 4\cos(\theta)} \end{aligned}$$

So the equation $y = 4x - 3$ in rectangular coordinates is equivalent to the equation $r = -\dfrac{3}{\sin(\theta) - 4\cos(\theta)}$ in polar coordinates.

Now let's discuss graphing functions in polar coordinates. Just as when we graph functions in rectangular coordinates, we can graph functions in polar coordinates by finding ordered pairs that satisfy the function. Ordered pairs in polar coordinates have the form (r, θ). **Notice that this convention for the polar ordered pair puts the *output* variable, r, first and the *input* variable, θ, second**. This is **different** from rectangular ordered pairs of the form (x, y) that have the *input* variable first and the *output* variable second.

EXAMPLE 3: Sketch the graph of the function $r = \theta$ on the polar coordinate plane.

SOLUTION:

CLICK HERE to see $r = \theta$ graphed by hand.

Note that if you want to graph this function on your calculator, you need to change the graphing **mode** of your calculator to ***POLAR***.

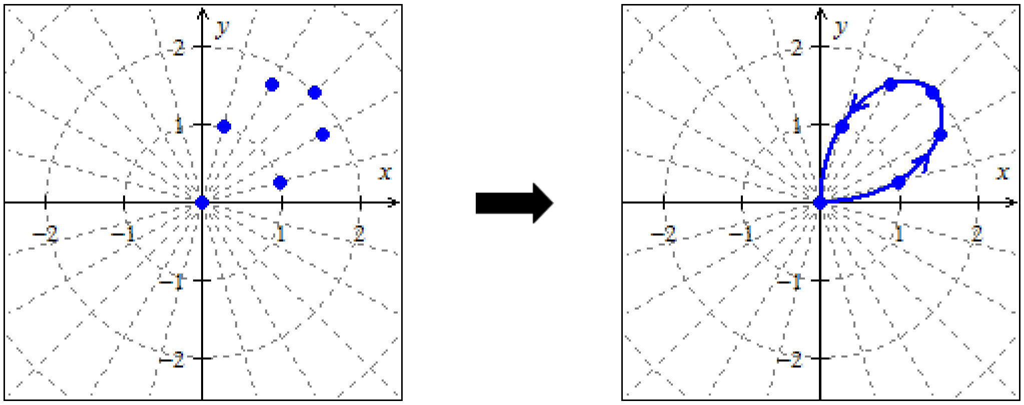
EXAMPLE 4: Sketch the graph of the $r = 2\sin(2\theta)$ on the polar coordinate plane.

SOLUTION:

To sketch the graph of $r = 2\sin(2\theta)$, we should find some ordered pairs (r, θ) that satisfy the function. The table below gives some values for $r = 2\sin(2\theta)$.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$r = 2\sin(2\theta)$	0	1	$\sqrt{3} \approx 1.7$	2	$\sqrt{3} \approx 1.7$	1	0
(r, θ) (approximately)	$(0, 0)$	$\left(1, \frac{\pi}{12}\right)$	$\left(1.7, \frac{\pi}{6}\right)$	$\left(2, \frac{\pi}{4}\right)$	$\left(1.7, \frac{\pi}{3}\right)$	$\left(1, \frac{5\pi}{12}\right)$	$\left(0, \frac{\pi}{2}\right)$

Let’s plot these points and connect them:



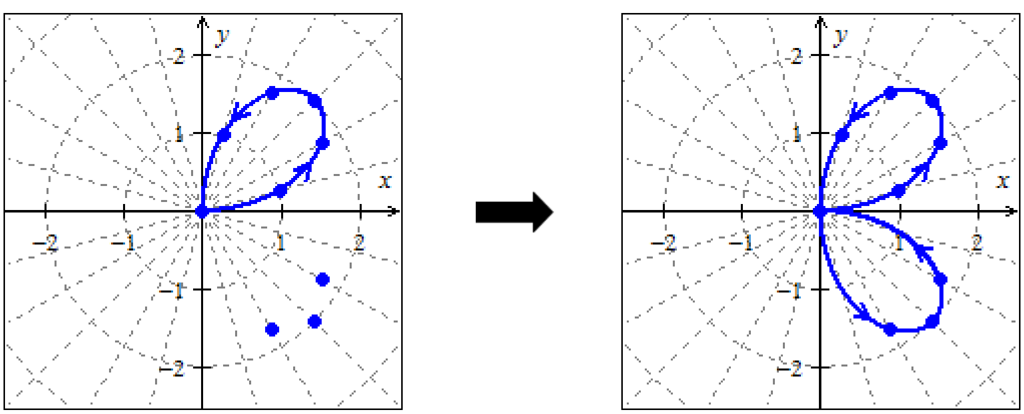
Points on the graph of $r = 2\sin(2\theta)$. A portion of the graph of $r = 2\sin(2\theta)$.

Figure 1

Now let’s find some more points on the graph of $r = 2\sin(2\theta)$.

θ	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$r = 2\sin(2\theta)$	$-\sqrt{3} \approx -1.7$	-2	$-\sqrt{3} \approx -1.7$	0
(r, θ) (approximately)	$\left(-1.7, \frac{2\pi}{3}\right)$	$\left(-2, \frac{3\pi}{4}\right)$	$\left(-1.7, \frac{5\pi}{6}\right)$	$(0, \pi)$

Let’s plot these points. Keep in mind that negative r-values are plotted in the opposite quadrant than the angle θ. Since all of these angles are in Quadrant II, we will end up plotting points in Quadrant IV.

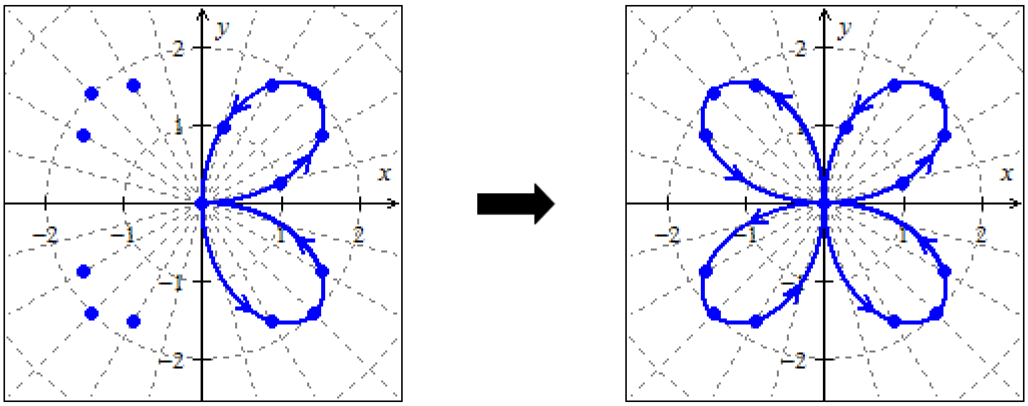


More points on the graph of $r = 2\sin(2\theta)$. A larger portion of the graph of $r = 2\sin(2\theta)$.

Figure 2

Now we can find a few more points and finish our graph of $r = 2\sin(2\theta)$; see Figure 3.

θ	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$r = 2\sin(2\theta)$	$\sqrt{3} \approx 1.7$	2	$\sqrt{3} \approx 1.7$	0	$-\sqrt{3} \approx -1.7$	-2	$-\sqrt{3} \approx -1.7$	0
(r, θ) (approximately)	$\left(1.7, \frac{7\pi}{6}\right)$	$\left(2, \frac{5\pi}{4}\right)$	$\left(1.7, \frac{4\pi}{3}\right)$	$\left(0, \frac{3\pi}{2}\right)$	$\left(-1.7, \frac{5\pi}{3}\right)$	$\left(-2, \frac{7\pi}{4}\right)$	$\left(-1.7, \frac{11\pi}{6}\right)$	$(0, 2\pi)$



More points on the graph of $r = 2\sin(2\theta)$.

The compete graph of $r = 2\sin(2\theta)$.

Figure 3

You should make sure you can get this graph of $r = 2\sin(2\theta)$ on your graphing calculator. Don't forget to change the graphing mode of your calculator to ***polar***.

EXAMPLE 5: Sketch the graph of the $r = 3$ on the polar coordinate plane.

SOLUTION:

Here, since $r = 3$, the distance from the origin is *always* 3 units. So no matter what θ is, $r = 3$. This gives us a **circle** of radius 3; see Figure 4.

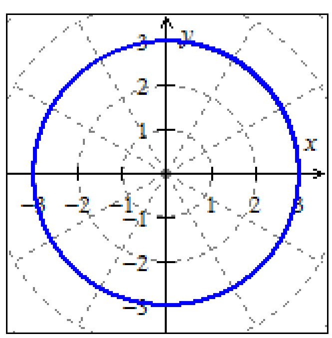
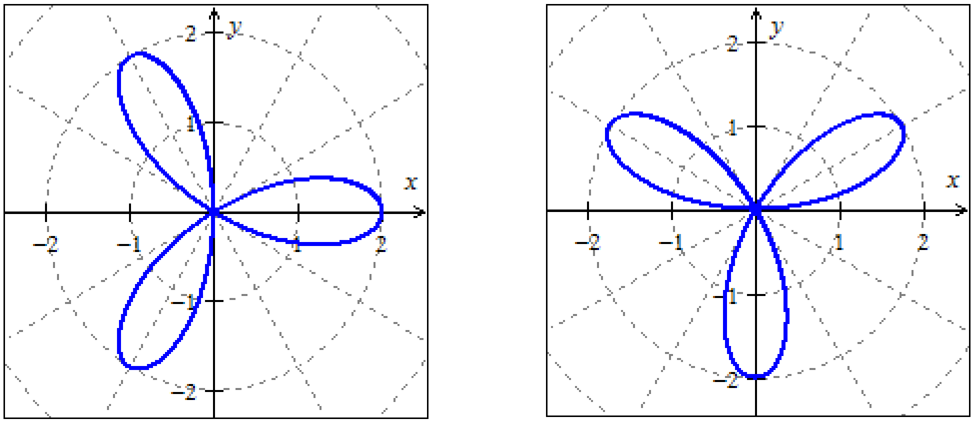


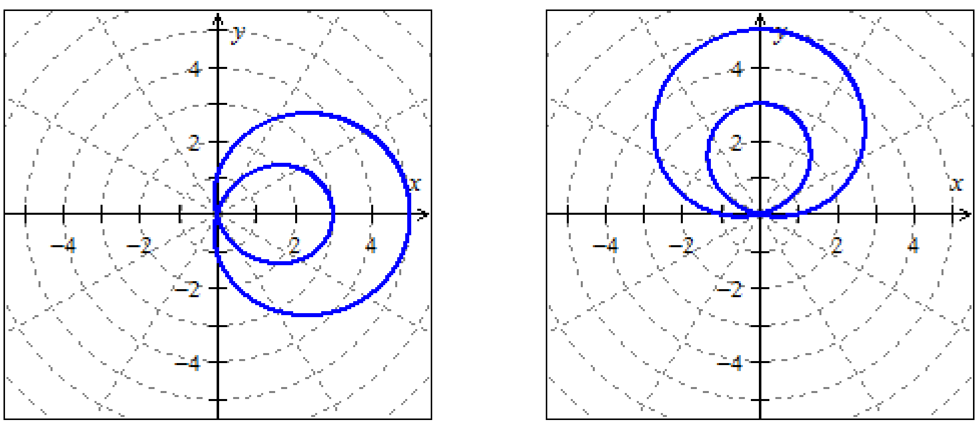
Figure 4: Graph of $r = 3$.

Below are the graphs of a few other functions defined via polar coordinates. You should graph them on your graphing calculator.



$r = 2\cos(3\theta)$ $r = 2\sin(3\theta)$

Figure 5: These graphs and the graph given in Figure 3 are called "**roses**".



$r = 1 + 4\cos(\theta)$ $r = 1 + 4\sin(\theta)$

Figure 6: These graphs are called "**limaçons**".