Section III: Polar Coordinates and Complex Numbers

Chapter 1: Introduction to Polar Coordinates

We are all comfortable using rectangular (i.e., Cartesian) coordinates to describe points on the plane. For example, we've plotted the point $P = \left(\sqrt{3}, 1\right)$ on the coordinate plane in Figure 1.

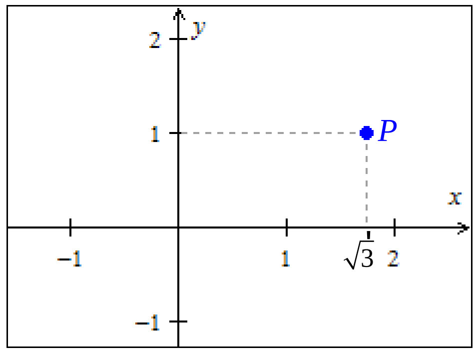

Figure 1: The point $P = \left(\sqrt{3}, 1\right)$ on the rectangular coordinate plane.

Instead of using these *rectangular* coordinates, we can use a *circular* coordinate system to describe points on the plane: **Polar Coordinates**. Ordered pairs in polar coordinates have form (r, θ) where r represents the point's distance from the origin and θ represents the angular displacement of the point with respect to the positive x-axis. Let's find the polar coordinates that describe P in Figure 1.

First let's find r, the distance from point P to the origin; in other words, we need to find the length of the segment labeled r in Figure 2:

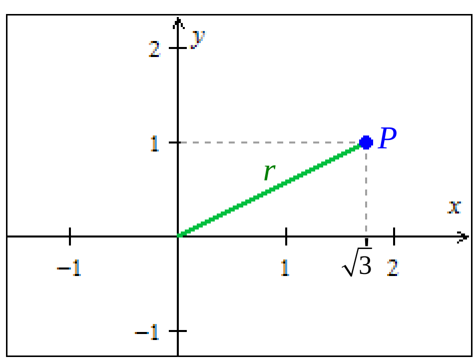

Figure 2

We can use the Pythagorean Theorem to find r :

$$\begin{aligned} r^2 &= \left(\sqrt{3}\right)^2 + (1)^2 \\ \Rightarrow r^2 &= 3 + 1 \\ \Rightarrow r^2 &= 4 \\ \Rightarrow r &= 2 \end{aligned}$$

Now we need to find the angle between the positive x-axis and the segment labeled r; this angle is labeled θ in Figure 3.

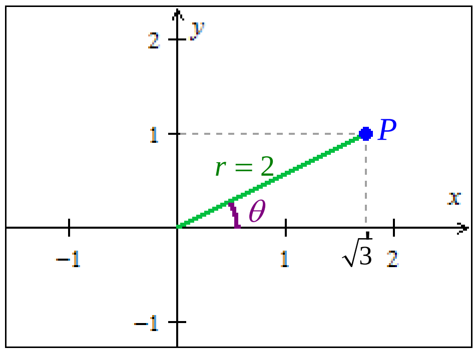

Figure 3

We can use the right triangle induced by the angle θ and the side r along with either sine or cosine to find the value of θ:

$$\begin{aligned} \sin(\theta) &= \tfrac{1}{2} \\ \Rightarrow \quad \theta &= \tfrac{\pi}{6} \end{aligned}$$

Thus, in polar coordinates, $P = \left(2, \frac{\pi}{6}\right)$. We've plotted the point P on the polar coordinate plane in Figure 4.

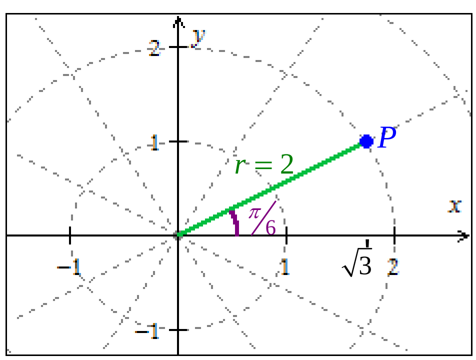

Figure 4: The point $P = \left(2, \frac{\pi}{6}\right)$ on the polar coordinate

EXAMPLE 1: Plot the point $A = \left(10, \frac{5\pi}{4}\right)$ on the polar coordinate plane and determine the rectangular coordinates of point A.

SOLUTION:

To plot the point $A = \left(10, \frac{5\pi}{4}\right)$ we need to recognize that polar ordered pairs have form (r, θ), so $A = \left(10, \frac{5\pi}{4}\right)$ implies that

$$r = 10 \quad \text{and} \quad \theta = \frac{5\pi}{4}.$$

We've plotted the point $A = \left(10, \frac{5\pi}{4}\right)$ on the polar coordinate plane in Figure 5.

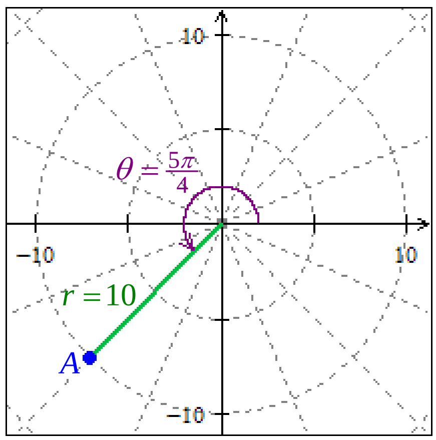

Figure 5

To find the rectangular coordinates of point A we can use the reference angle for θ, which is $\frac{\pi}{4}$, and the induced right triangle; see Figure 6.

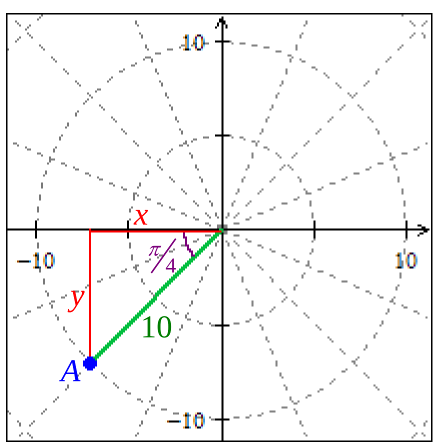

Figure 6

Using the triangle in Figure 6, we can see that

$$\begin{aligned} &\cos\left(\tfrac{\pi}{4}\right) = \tfrac{|x|}{10} \\ \Rightarrow\quad &|x| = 10\cos\left(\tfrac{\pi}{4}\right) \\ \Rightarrow\quad &|x| = 10\cdot\tfrac{\sqrt{2}}{2} \\ &\quad = 5\sqrt{2} \end{aligned} \qquad \text{and} \qquad \begin{aligned} &\sin\left(\tfrac{\pi}{4}\right) = \tfrac{|y|}{10} \\ \Rightarrow\quad &|y| = 10\sin\left(\tfrac{\pi}{4}\right) \\ \Rightarrow\quad &|y| = 10\cdot\tfrac{\sqrt{2}}{2} \\ &\quad = 5\sqrt{2} \end{aligned}$$

Since point A is in Quadrant III, we know that both x and y are negative. Thus, the rectangular coordinates of point A are $\left(-5\sqrt{2},\, -5\sqrt{2}\right)$.

EXAMPLE 2: Find the rectangular coordinates of a generic point $P = (r,\, \theta)$ on the polar coordinate plane.

SOLUTION:

In Figure 7, we've pointed P plotted in the polar plane.

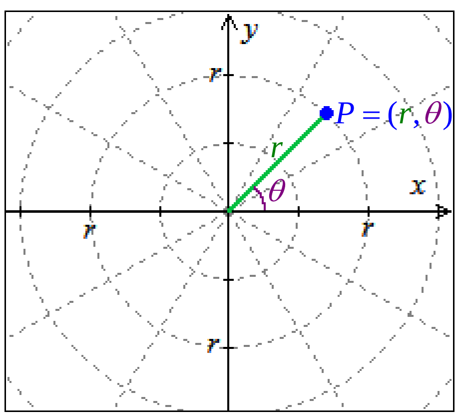

Figure 7

We can construct a right triangle and use trigonometry to obtain expressions for the horizontal and vertical coordinates of point P; see Figure 8 below.

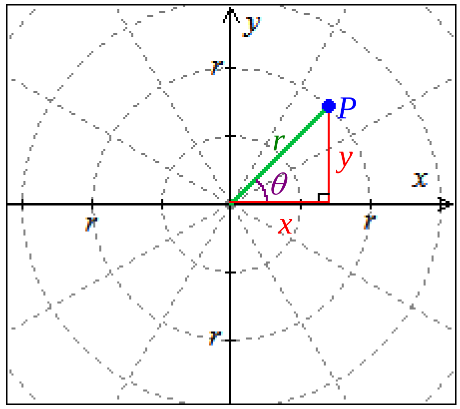

Figure 8

Based on the triangle in Figure 8, we can see that

$$\cos(\theta) = \frac{x}{r} \quad \text{and} \quad \sin(\theta) = \frac{y}{r}$$

$$\Rightarrow \quad x = r\cos(\theta) \qquad \Rightarrow \quad y = r\sin(\theta)$$

Thus, if $P = (r, \theta)$ is represents a point on the polar coordinate plane, then the rectangular coordinates of P are $(x, y) = (r\cos(\theta), r\sin(\theta))$. (Notice that we observed essentially the same fact in Section I: Chapter 3.) We can use what we've discovered to translate polar coordinates into rectangular coordinates.

> The polar coordinates (r, θ) are equivalent to the rectangular coordinates
>
> $$(x, y) = (r\cos(\theta), r\sin(\theta)).$$

Key Point:

Polar and rectangular ordered pairs cannot be set equal to each other. When ordered pairs are described as being equal, it means that they have the same coordinates so we can write something like $(0.75, 0.5) = \left(\frac{3}{4}, \frac{1}{2}\right)$ since $0.75 = \frac{3}{4}$ and $0.5 = \frac{1}{2}$ but we can't write $\left(10, \frac{5\pi}{4}\right) = \left(-5\sqrt{2}, -5\sqrt{2}\right)$ (from Example 1) since $10 \neq -5\sqrt{2}$ and $\frac{5\pi}{4} \neq -5\sqrt{2}$. In order to communicate that rectangular ordered pairs and polar ordered pairs describe the same location, we need to compose sentences like, "The rectangular ordered pair $\left(-5\sqrt{2}, -5\sqrt{2}\right)$ is equivalent to the polar ordered pair $\left(10, \frac{5\pi}{4}\right)$."

EXAMPLE 3: Plot the point $B = \left(-4, \frac{2\pi}{3}\right)$ on the polar coordinate plane and find the rectangular coordinates of the point.

SOLUTION:

To plot the point $B = \left(-4, \frac{2\pi}{3}\right)$ we need to recognize that polar ordered pairs have form (r, θ), so $B = \left(-4, \frac{2\pi}{3}\right)$ implies that

$$r = -4 \text{ and } \theta = \frac{2\pi}{3}.$$

Here, r is negative. This means that when we get to the terminal side of $\theta = \frac{2\pi}{3}$, instead of going "forward" 4 units into Quadrant II, we need to go "backwards" 4 units into Quadrant IV; see Figure 9.

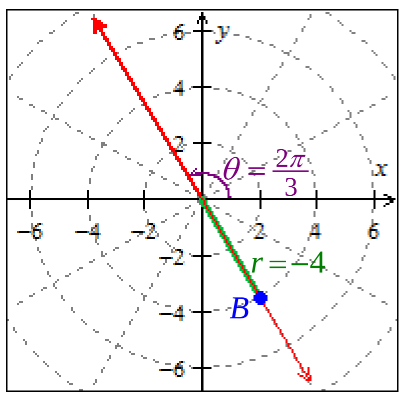

Figure 9

To find the rectangular coordinates of point B, we can use the conversion equations we derived in the previous example.

$$\begin{aligned} x &= r\cos(\theta) \\ &= -4\cos\left(\tfrac{2\pi}{3}\right) \\ &= -4\cdot\left(-\tfrac{1}{2}\right) \\ &= 2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} y &= r\sin(\theta) \\ &= -4\sin\left(\tfrac{2\pi}{3}\right) \\ &= -4\cdot\left(\tfrac{\sqrt{3}}{2}\right) \\ &= -2\sqrt{3} \end{aligned}$$

Thus, the rectangular coordinates of B are $(x, y) = \left(2, -2\sqrt{3}\right)$.