Extra Practice for Section II: Chapter 5

These problems involve some identities that you don't need to memorize since they are listed on the "Identities and Formulas Reference Sheet" that I'll include with your Final Exam. You are encouraged to consult the "Identities and Formulas Reference Sheet (provided here, at the end of this document) while you work on these problems so that you get familiar with using it.

1. Suppose that $\sin(\alpha) = -\frac{\sqrt{65}}{9}$ and $\pi < \alpha < \frac{3\pi}{2}$. Use a **double-angle** or **half-angle identity** to calculate the *exact value* of each of the following:

a. $\sin(2\alpha)$. [Hint: first find $\cos(\alpha)$] Click here to see the solution to **1.a.**

b. $\cos(2\alpha)$. Click here to see the solution to **1.b.**

c. $\sin\left(\frac{\alpha}{2}\right)$. Click here to see the solution to **1.c.**

d. $\cos\left(\frac{\alpha}{2}\right)$. Click here to see the solution to **1.d.**

2. Suppose that $\cos(\phi) = \frac{7}{10}$ and $\frac{3\pi}{2} < \phi < 2\pi$. Use a **double-angle** or **half-angle identity** to calculate the *exact value* of each of the following:

a. $\sin(2\phi)$. [Hint: first find $\sin(\phi)$] Click here to see the solution to **2.a.**

b. $\cos(2\phi)$. Click here to see the solution to **2.b.**

c. $\sin\left(\frac{\phi}{2}\right)$. Click here to see the solution to **2.c.**

d. $\cos\left(\frac{\phi}{2}\right)$. Click here to see the solution to **2.d.**

3. Prove this identity: $\dfrac{1-\cos(2t)}{\sin(2t)} = \tan(t)$.

Click here to see the solution to **3.**

Solution to **1.a.**

1.a. $\sin(2\alpha)$. [Hint: first find $\cos(\alpha)$]

Since the double-angle identity for sine involves $\cos(\alpha)$ let's first find this using the Pythagorean identity:

$$\begin{aligned}
& \sin^2(\alpha) + \cos^2(\alpha) = 1 \\
\Rightarrow\ & \left(-\tfrac{\sqrt{65}}{9}\right)^2 + \cos^2(\alpha) = 1 \qquad \text{(since } \sin(\alpha) = -\tfrac{\sqrt{65}}{9}\text{)} \\
\Rightarrow\ & \cos^2(\alpha) = 1 - \tfrac{65}{81} \\
\Rightarrow\ & \cos^2(\alpha) = \tfrac{16}{81} \\
\Rightarrow\ & \cos(\alpha) = -\tfrac{4}{9} \qquad \text{(note that we take the negative square root since cosine is negative in the 3rd quadrant)}
\end{aligned}$$

Thus,

$$\begin{aligned}
\sin(2\alpha) &= 2\sin(\alpha)\cos(\alpha) \\
&= 2\left(-\tfrac{\sqrt{65}}{9}\right)\left(-\tfrac{4}{9}\right) \qquad \text{(since } \sin(\alpha) = -\tfrac{\sqrt{65}}{9} \text{ and } \cos(\alpha) = -\tfrac{4}{9}\text{)} \\
&= \tfrac{8\sqrt{65}}{81}
\end{aligned}$$

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Solution to **1.b.**

1.b. $\cos(2\alpha)$.

We could use any one of the three double-angle identities for cosine but we'll choose the identity that only involves sine since we are given the sine value in the question. (If we use given info rather than info that we've discovered in **1.a.**, we avoid the possibility of using incorrect info if we made mistakes in **1.a.**)

$$\begin{aligned}
\cos(2\alpha) &= 1 - 2\sin^2(\alpha) \\
&= 1 - 2\cdot\left(-\tfrac{\sqrt{65}}{9}\right)^2 \qquad \text{(since } \sin(\alpha) = -\tfrac{\sqrt{65}}{9}\text{)} \\
&= 1 - 2\cdot\tfrac{65}{81} \\
&= \tfrac{81}{81} - \tfrac{130}{81} \\
&= -\tfrac{49}{81}
\end{aligned}$$

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Solution to **1.c.**

1.c. $\sin\left(\frac{\alpha}{2}\right)$.

The half-angle identity for sine is $\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}}$, and the "$\pm$" in the identity suggests that we need to determine which sign is correct in this case. Since

$$\begin{aligned} &\pi < \alpha < \frac{3\pi}{2} \\ \Rightarrow\ &\frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{2\cdot 2} \\ \Rightarrow\ &\frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4}, \end{aligned}$$

we see that $\frac{\alpha}{2}$ is in the 2nd quadrant so $\sin\left(\frac{\alpha}{2}\right) > 0$, so we need the positive value:

$$\begin{aligned} \sin\left(\frac{\alpha}{2}\right) &= +\sqrt{\frac{1-\cos(\alpha)}{2}} \\ &= \sqrt{\frac{1-\left(-4/9\right)}{2}} \qquad \text{(since } \cos(\alpha) = -4/9\text{)} \\ &= \sqrt{\frac{13}{9}\cdot\frac{1}{2}} \\ &= \sqrt{\frac{13}{18}} = \frac{\sqrt{13}}{\sqrt{9\cdot 2}} = \frac{\sqrt{13}}{3\sqrt{2}} = \frac{\sqrt{26}}{6} \end{aligned}$$

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Solution to **1.d.**

1.d. $\cos\left(\frac{\alpha}{2}\right)$.

The half-angle identity for cosine is $\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos(\alpha)}{2}}$, and the "$\pm$" in the identity suggests that we need to determine which sign is correct in this case. In **1.c.**, we determined that $\frac{\alpha}{2}$ is in quadrant 2; thus, $\cos\left(\frac{\alpha}{2}\right) < 0$ and we need the negative value:

$$\begin{aligned} \cos\left(\frac{\alpha}{2}\right) &= -\sqrt{\frac{1+\cos(\alpha)}{2}} \\ &= -\sqrt{\frac{1+\left(-4/9\right)}{2}} \qquad \text{(since } \cos(\alpha) = -4/9\text{)} \\ &= -\sqrt{\frac{5}{9}\cdot\frac{1}{2}} \\ &= -\sqrt{\frac{5}{18}} = -\frac{\sqrt{5}}{3\sqrt{2}} = -\frac{\sqrt{10}}{6} \end{aligned}$$

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Solution to **2.a.**

2.a. $\sin(2\phi)$. [Hint: first find $\cos(\phi)$]

Since the double-angle identity for sine involves $\cos(\phi)$ let's first find this using the Pythagorean identity:

$$\sin^2(\phi) + \cos^2(\phi) = 1$$

$$\Rightarrow \sin^2(\phi) + \left(\tfrac{7}{10}\right)^2 = 1 \qquad \text{(since } \cos(\phi) = \tfrac{7}{10}\text{)}$$

$$\Rightarrow \sin^2(\phi) = 1 - \tfrac{49}{100}$$

$$\Rightarrow \sin(\phi) = -\tfrac{\sqrt{51}}{10} \qquad \text{(note that we take the negative square root since sine is negative in the 4th quadrant)}$$

Thus,

$$\begin{aligned} \sin(2\phi) &= 2\sin(\phi)\cos(\phi) \\ &= 2\left(-\tfrac{\sqrt{51}}{10}\right)\left(\tfrac{7}{10}\right) \qquad \text{(since } \sin(\phi) = -\tfrac{\sqrt{51}}{10} \text{ and } \cos(\phi) = \tfrac{7}{10}\text{)} \\ &= -\tfrac{14\sqrt{51}}{100} \end{aligned}$$

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Solution to **2.b.**

2.b. $\cos(2\phi)$.

We could use any one of the three double-angle identities for cosine but we'll choose the identity that only involves cosine since we are given the cosine value in the question. (If we use given info rather than info that we've discovered in **2.a.**, we avoid the possibility of using incorrect info if we made mistakes in **2.a.**)

$$\begin{aligned} \cos(2\phi) &= 2\cos^2(\phi) - 1 \\ &= 2\cdot\left(\tfrac{7}{10}\right)^2 - 1 \qquad \text{(since } \cos(\phi) = \tfrac{7}{10}\text{)} \\ &= 2\cdot\tfrac{49}{100} - 1 \\ &= \tfrac{98}{100} - \tfrac{100}{100} \\ &= -\tfrac{2}{100} \\ &= -\tfrac{1}{50} \end{aligned}$$

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Solution to **2.c.**

2.c. $\sin\left(\frac{\phi}{2}\right)$.

The half-angle identity for sine is $\sin\left(\frac{\phi}{2}\right) = \pm\sqrt{\frac{1-\cos(\phi)}{2}}$, and the "$\pm$" in the identity suggests that we need to determine which sign is correct in this case. Since

$$\begin{aligned} &\frac{3\pi}{2} < \phi < 2\pi \\ \Rightarrow\ &\frac{3\pi}{2\cdot 2} < \frac{\phi}{2} < \frac{2\pi}{2} \\ \Rightarrow\ &\frac{3\pi}{4} < \frac{\phi}{2} < \pi, \end{aligned}$$

we see that $\frac{\phi}{2}$ is in the 2nd quadrant so $\sin\left(\frac{\phi}{2}\right) > 0$; thus, we'll use the positive value:

$$\begin{aligned} \sin\left(\frac{\phi}{2}\right) &= +\sqrt{\frac{1-\cos(\phi)}{2}} \\ &= \sqrt{\frac{1-7/10}{2}} \qquad \text{(since } \cos(\phi) = 7/10\text{)} \\ &= \sqrt{\frac{3}{10}\cdot\frac{1}{2}} \\ &= \sqrt{\frac{3}{20}} = \frac{\sqrt{3}}{\sqrt{20}} = \frac{\sqrt{60}}{20} \end{aligned}$$

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Solution to **2.d.**

2.d. $\cos\left(\frac{\phi}{2}\right)$.

The half-angle identity for cosine is $\cos\left(\frac{\phi}{2}\right) = \pm\sqrt{\frac{1+\cos(\phi)}{2}}$; as discussed above we need to determine which sign is correct for the value that we are computing. In **2.c.**, we determined that $\frac{\phi}{2}$ is in quadrant 2, so $\cos\left(\frac{\phi}{2}\right) < 0$, so we'll use the negative value:

$$\begin{aligned} \cos\left(\frac{\phi}{2}\right) &= -\sqrt{\frac{1+\cos(\phi)}{2}} \\ &= -\sqrt{\frac{1+7/10}{2}} \qquad \text{(since } \cos(\phi) = 7/10\text{)} \\ &= -\sqrt{\frac{17}{10}\cdot\frac{1}{2}} \\ &= -\sqrt{\frac{17}{20}} = -\frac{\sqrt{17}}{2\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = -\frac{\sqrt{85}}{2\cdot 5} = -\frac{\sqrt{85}}{10} \end{aligned}$$

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Solution to **3.**

3. Prove this identity: $\dfrac{1-\cos(2t)}{\sin(2t)} = \tan(t)$.

$$\begin{aligned}
\frac{1-\cos(2t)}{\sin(2t)} &= \frac{1-\left(1-2\sin^2(t)\right)}{2\sin(t)\cos(t)} \\
&= \frac{2\sin^2(t)}{2\sin(t)\cos(t)} \\
&= \frac{\cancel{2}\sin^{\cancel{2}}(t)}{\cancel{2}\,\cancel{\sin(t)}\cos(t)} \\
&= \frac{\sin(t)}{\cos(t)} \\
&= \tan(t)
\end{aligned}$$

(since $\cos(2t) = 1-2\sin^2(t)$
and $\sin(2t) = 2\sin(t)\cos(t)$)

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MTH 112: Identities and Formulas Reference Sheet

This sheet will be provided to students during the Final Exam.

Law of Sines

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

Double Angle Identities

$$\sin(2A) = 2\sin(A)\cos(A)$$
$$\cos(2A) = 1 - 2\sin^2(A)$$
$$\cos(2A) = 2\cos^2(A) - 1$$
$$\cos(2A) = \cos^2(A) - \sin^2(A)$$
$$\tan(2A) = \frac{2\tan(A)}{1-\tan^2(A)}$$

Sum and Difference Identities

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$
$$\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$
$$\tan(A-B) = \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$$

Half Angle Identities

$$\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{2}}$$
$$\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos(A)}{2}}$$
$$\tan\left(\frac{A}{2}\right) = \frac{1-\cos(A)}{\sin(A)}$$

Product-to-Sum Identities

$$\sin(A)\sin(B) = \tfrac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$
$$\cos(A)\cos(B) = \tfrac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$
$$\sin(A)\cos(B) = \tfrac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

Conic Sections: Ellipses

IMPLICIT EQUATION: $1 = \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2}$

PARAMETRIC SYSTEM: $\begin{cases} x = a\cos(t) + h \\ y = b\sin(t) + k \end{cases}$

Sum of Sine and Cosine Identity

$$A_1\sin(\omega t) + A_2\cos(\omega t) = A\sin(\omega t + \phi)$$

where $A = \sqrt{A_1 + A_2}$ and $\tan(\phi) = \frac{A_2}{A_1}$, and

ϕ satisfies $\cos(\phi) = \frac{A_1}{A}$ and $\sin(\phi) = \frac{A_2}{A}$

Dot Product

$$\vec{v}\cdot\vec{w} = a_1a_2 + b_1b_2$$
$$\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos(\theta)$$

Euler's Formula

$$re^{i\cdot\theta} = r\cos(\theta) + r\sin(\theta)\cdot i$$

Sum-to-Product Identities

$$\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\sin(A) - \sin(B) = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$
$$\cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\cos(A) - \cos(B) = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

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