Extra Practice for Section I: Chapter 6

You should complete all of these problems **without a calculator** in order to prepare for the Midterm which is a no-calculator exam.

1. Find the exact value of each of the following expressions; do not use a calculator. Be sure to use proper notation to ***directly communicate*** what the given expressions equal.

a. $\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

b. $\cos^{-1}\left(\cos\left(\frac{5\pi}{3}\right)\right)$

c. $\cos^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$

d. $\sin\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

e. $\sin^{-1}\left(\sin\left(\frac{7\pi}{8}\right)\right)$

f. $\cos^{-1}\left(\cos\left(\frac{6\pi}{5}\right)\right)$

Click here to see the solution to **1.**

2. Find *all* of the solutions to the equations below; provide *exact* solutions.

a. $2\sin(x) - \sqrt{3} = 0$ Click here to see the solution to **2.a.**

b. $5 + 4\cos(2\theta) = 1$ Click here to see the solution to **2.b.**

c. $16 - 24\sin(4t) = 4$ Click here to see the solution to **2.c.**

d. $6\sqrt{2}\cos(3\alpha) + 10 = 4$ Click here to see the solution to **2.d.**

3. Find the solutions to the equations below on the interval $[0, 2\pi)$; provide *exact* solutions.

a. $5 + 4\cos(2\theta) = 1$ Click here to see the solution to **3.a.**

b. $6\sqrt{2}\cos(3\alpha) + 10 = 4$ Click here to see the solution to **3.b.**

Solution to **1.**

1. Find the exact value of each of the following expressions; do not use a calculator. Be sure to use proper notation to ***directly communicate*** what the given expressions equal.

a. $\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

$$\begin{aligned}\sin\left(\sin^{-1}\left(\tfrac{\sqrt{3}}{2}\right)\right) &= \sin\left(\tfrac{\pi}{3}\right) && \text{(since } \tfrac{\sqrt{3}}{2} = \sin\left(\tfrac{\pi}{3}\right) \text{ and } -\tfrac{\pi}{2} \le \tfrac{\pi}{3} \le \tfrac{\pi}{2}\text{)}\\ &= \tfrac{\sqrt{3}}{2} && \text{(since } \sin\left(\tfrac{\pi}{3}\right) = \tfrac{\sqrt{3}}{2}\text{)}\end{aligned}$$

b. $\cos^{-1}\left(\cos\left(\frac{5\pi}{3}\right)\right)$

$$\begin{aligned}\cos^{-1}\left(\cos\left(\tfrac{5\pi}{3}\right)\right) &= \cos^{-1}\left(\tfrac{1}{2}\right) && \text{(since } \cos\left(\tfrac{5\pi}{3}\right) = \tfrac{1}{2}\text{)}\\ &= \tfrac{\pi}{3} && \text{(since } \cos^{-1}\left(\tfrac{1}{2}\right) = \tfrac{\pi}{3} \text{ and } 0 \le \tfrac{\pi}{3} \le \pi\text{)}\end{aligned}$$

c. $\cos^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$

$$\begin{aligned}\cos^{-1}\left(\tan\left(\tfrac{3\pi}{4}\right)\right) &= \cos^{-1}(-1) && \text{(since } \tan\left(\tfrac{3\pi}{4}\right) = -1\text{)}\\ &= \pi && \text{(since } \cos^{-1}(-1) = \pi \text{ and } 0 \le \pi \le \pi\text{)}\end{aligned}$$

d. $\sin\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

$$\begin{aligned}\sin\left(\cos^{-1}\left(-\tfrac{\sqrt{3}}{2}\right)\right) &= \sin\left(\tfrac{5\pi}{6}\right) && \text{(since } -\tfrac{\sqrt{3}}{2} = \cos\left(\tfrac{5\pi}{6}\right) \text{ and } 0 \le \tfrac{5\pi}{6} \le \pi\text{)}\\ &= \tfrac{1}{2} && \text{(since } \sin\left(\tfrac{5\pi}{6}\right) = \tfrac{1}{2}\text{)}\end{aligned}$$

e. $\sin^{-1}\left(\sin\left(\frac{7\pi}{8}\right)\right)$

$\frac{7\pi}{8}$ isn't a "friendly angle" so we aren't familiar with its sine value and cannot use the same "mechanical" step-by-step approach we used in parts (a) – (d). Instead, we are forced to rely on our conceptual understanding of the sine function:

$$\begin{aligned}\sin^{-1}\left(\sin\left(\frac{7\pi}{8}\right)\right) &= \sin^{-1}\left(\sin\left(\frac{\pi}{8}\right)\right) && \left(\text{since } \sin\left(\frac{7\pi}{8}\right) = \sin\left(\pi - \frac{7\pi}{8}\right)\right) \\ &= \frac{\pi}{8} && \left(\text{since } -\frac{\pi}{2} \le \frac{\pi}{8} \le \frac{\pi}{2}\right)\end{aligned}$$

f. $\cos^{-1}\left(\cos\left(\frac{6\pi}{5}\right)\right)$

$\frac{6\pi}{5}$ isn't a "friendly angle" so we aren't familiar with its cosine value and cannot use the same "mechanical" step-by-step approach we used in parts (a) – (d). Instead, we are forced to rely on our conceptual understanding of the cosine function:

$$\begin{aligned}\cos^{-1}\left(\cos\left(\frac{6\pi}{5}\right)\right) &= \cos^{-1}\left(\cos\left(\frac{4\pi}{5}\right)\right) && \left(\text{since } \cos\left(\frac{6\pi}{5}\right) = \cos\left(-\frac{6\pi}{5}\right) \text{ and } -\frac{6\pi}{5} \text{ is coterminal with } \frac{4\pi}{5}\right) \\ &= \frac{4\pi}{5} && \left(\text{since } 0 \le \frac{4\pi}{5} \le \pi\right)\end{aligned}$$

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2. Find the exact value of each of the following expressions; do not use a calculator. Be sure to use proper notation to ***directly communicate*** what the given expressions equal.

Solution to **2.a.**

a. $2\sin(x) - \sqrt{3} = 0$

$$2\sin(x) - \sqrt{3} = 0$$

$$\Rightarrow \quad 2\sin(x) = \sqrt{3}$$

$$\Rightarrow \quad \sin(x) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad x = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + 2k\pi \quad \text{or} \quad x = \pi - \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + 2k\pi, \ \ k \in \mathbb{Z}$$

$$\Rightarrow \quad x = \frac{\pi}{3} + 2k\pi \quad \text{or} \quad x = \pi - \frac{\pi}{3} + 2k\pi, \ \ k \in \mathbb{Z}$$

$$\Rightarrow \quad x = \frac{\pi}{3} + 2k\pi \quad \text{or} \quad x = \frac{2\pi}{3} + 2k\pi, \ \ k \in \mathbb{Z}$$

(Note that you might choose to omit the step indicated by a red arrow ($\Rightarrow$) and, possibly, the step indicated by a pink arrow ($\Rightarrow$) and, instead, utilize your awareness of these two facts: $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$.)

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Solution to **2.b.**

b. $5 + 4\cos(2\theta) = 1$

$$5 + 4\cos(2\theta) = 1$$

$$\Rightarrow \quad 4\cos(2\theta) = -4$$

$$\Rightarrow \quad \cos(2\theta) = -1$$

$$\Rightarrow \quad 2\theta = \cos^{-1}(-1) + 2k\pi, \quad k \in \mathbb{Z}$$

(there's only one "family" of solutions since the cosine function achieves the output -1 only once in each period)

$$\Rightarrow \quad 2\theta = \pi + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad \frac{2\theta}{2} = \frac{\pi}{2} + \frac{2k\pi}{2}, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad \theta = \tfrac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

(Note that you might choose to omit the step indicated by a red arrow ($\Rightarrow$) and, instead, utilize your awareness of the fact that $\cos(\pi) = -1$.)

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Solution to **2.c.**

c. $16 - 24\sin(4t) = 4$

$$15 - 24\sin(4t) = 3$$

$$\Rightarrow \quad -24\sin(4t) = -12$$

$$\Rightarrow \quad \sin(4t) = \tfrac{-12}{-24} = \tfrac{1}{2}$$

$$\Rightarrow \quad 4t = \sin^{-1}\left(\tfrac{1}{2}\right) + 2k\pi \quad \text{or} \quad 4t = \pi - \sin^{-1}\left(\tfrac{1}{2}\right) + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad 4t = \tfrac{\pi}{6} + 2k\pi \quad \text{or} \quad 4t = \pi - \tfrac{\pi}{6} + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad \frac{4t}{4} = \frac{\frac{\pi}{6}}{4} + \frac{2k\pi}{4} \quad \text{or} \quad \frac{4t}{4} = \frac{\frac{5\pi}{6}}{4} + \frac{2k\pi}{4}, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad t = \tfrac{\pi}{24} + \tfrac{k\pi}{2} \quad \text{or} \quad t = \tfrac{5\pi}{24} + \tfrac{k\pi}{2}, \quad k \in \mathbb{Z}$$

(Note that you might choose to omit the step indicated by a red arrow ($\Rightarrow$) and, possibly, the step indicated by a pink arrow ($\Rightarrow$) and, instead, utilize your awareness of these two facts: $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$.)

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Solution to **2.d.**

d. $6\sqrt{2}\cos(3\alpha) + 10 = 4$

$$
\begin{aligned}
& 6\sqrt{2}\cos(3\alpha) + 10 = 4 \\
\Rightarrow \quad & 6\sqrt{2}\cos(3\alpha) = -6 \\
\Rightarrow \quad & \cos(3\alpha) = \frac{-6}{6\sqrt{2}} = -\frac{1}{\sqrt{2}} \\
\Rightarrow \quad & 3\alpha = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) + 2k\pi \quad \text{or} \quad 3\alpha = -\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) + 2k\pi, \quad k \in \mathbb{Z} \\
\Rightarrow \quad & 3\alpha = \frac{3\pi}{4} + 2k\pi \quad \text{or} \quad 3\alpha = -\frac{3\pi}{4} + 2k\pi, \quad k \in \mathbb{Z} \\
\Rightarrow \quad & \frac{3\alpha}{3} = \frac{\frac{3\pi}{4}}{3} + \frac{2k\pi}{3} \quad \text{or} \quad \frac{3\alpha}{3} = \frac{-\frac{3\pi}{4}}{3} + \frac{2k\pi}{3}, \quad k \in \mathbb{Z} \\
\Rightarrow \quad & \alpha = \frac{3\pi}{4\cdot 3} + \frac{2k\pi}{3} \quad \text{or} \quad \alpha = -\frac{3\pi}{4\cdot 3} + \frac{2k\pi}{3}, \quad k \in \mathbb{Z} \\
\Rightarrow \quad & \alpha = \frac{\pi}{4} + \frac{2k\pi}{3} \quad \text{or} \quad \alpha = -\frac{\pi}{4} + \frac{2k\pi}{3}, \quad k \in \mathbb{Z}
\end{aligned}
$$

(Note that you might choose to omit the step indicated by a red arrow ($\Rightarrow$) and, possibly, the step indicated by a pink arrow ($\Rightarrow$) and, instead, utilize your awareness of these two facts: $\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ and $\cos\left(-\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$, although using "awareness" might inspire you to use "$\frac{5\pi}{4}$" instead of "$-\frac{3\pi}{4}$" for the second family of solutions, which will to lead you to "$\alpha = \frac{5\pi}{12} + \frac{2k\pi}{3}$" instead of "$\alpha = -\frac{\pi}{4} + \frac{2k\pi}{3}$" for the second family of solutions.)

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3. Find the solutions to the equations below on the interval $[0,\ 2\pi)$; provide *exact* solutions.

Solution to **3.a.**

a. $5+4\cos(2\theta)=1$

In #2b we determined that the solutions to this equation are represented by $\theta=\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}$. We need to substitute particular values of k in order to determine which solutions fall on the interval $[0,\ 2\pi)$:

$k=-1$: $\theta=\frac{\pi}{2}+(-1)\cdot\pi=-\frac{\pi}{2}<0$ so $-\frac{\pi}{2}\notin[0,\ 2\pi)$.

We could try smaller values of k but it should be clear that since $k=-1$ produced a value of θ that's too small, smaller values of k will produces even smaller values of θ so they won't produce solutions in the given interval so there's no need in trying smaller values of k.

$k=0$: $\theta=\frac{\pi}{2}+0\cdot\pi=\frac{\pi}{2}\in[0,\ 2\pi)$, so $\frac{\pi}{2}$ is a solution in the given interval.

$k=1$: $\theta=\frac{\pi}{2}+1\cdot\pi=\frac{3\pi}{2}\in[0,\ 2\pi)$, so $\frac{3\pi}{2}$ is a solution in the given interval.

$k=2$: $\theta=\frac{\pi}{2}+2\cdot\pi=\frac{5\pi}{2}>2\pi$ so $\frac{5\pi}{2}\notin[0,\ 2\pi)$.

We could try larger values of k but it should be clear that since $k=2$ produced a value of θ that's too large, larger values of k will produces even larger values of θ, so they won't produce solutions in the given interval so there's no need in trying larger values of k.

Thus, the solution set to $5+4\cos(2\theta)=1$ on the interval $[0,\ 2\pi)$ is $\left\{\frac{\pi}{2},\ \frac{3\pi}{2}\right\}$

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Solution to **3.b.**

b. $6\sqrt{2}\cos(3\alpha)+10=4$

In #2d we determined that the solutions to this equation are represented by "$\alpha=\frac{\pi}{4}+\frac{2k\pi}{3}$ or $\alpha=-\frac{\pi}{4}+\frac{2k\pi}{3},\ k\in\mathbb{Z}$." We need to substitute particular values of k into these equations to determine which solutions fall on the interval $[0,\,2\pi)$:

$k=-1$: $\alpha=\frac{\pi}{4}+\frac{2(-1)\pi}{3}=\frac{3\pi}{12}-\frac{8\pi}{12}=-\frac{5\pi}{12}$ or $\alpha=-\frac{\pi}{4}+\frac{2(-1)\pi}{3}=-\frac{3\pi}{12}-\frac{8\pi}{12}=-\frac{11\pi}{12}$

Since both of these values are negative, they aren't in the interval $[0,\,2\pi)$. We could try smaller values of k but it should be clear that since $k=-1$ resulted in a values of α that are too small, smaller values of k will produces even smaller values of α so they won't produce solutions in the given interval so there's no need in trying smaller values of k.

$k=0$: $\alpha=\frac{\pi}{4}+\frac{2\cdot0\cdot\pi}{3}=\frac{\pi}{4}$ or $\alpha=-\frac{\pi}{4}+\frac{2\cdot0\cdot\pi}{3}=-\frac{\pi}{4}$

Only $\frac{\pi}{4}$ is in the given interval.

$k=1$: $\alpha=\frac{\pi}{4}+\frac{2\cdot1\cdot\pi}{3}=\frac{3\pi}{14}+\frac{8\pi}{12}=\frac{11\pi}{12}$ or $\alpha=-\frac{\pi}{8}+\frac{2\cdot1\cdot\pi}{3}=-\frac{3\pi}{12}+\frac{8\pi}{12}=\frac{5\pi}{12}$

Both $\frac{11\pi}{12}$ and $\frac{5\pi}{12}$ are in the given interval.

$k=2$: $\alpha=\frac{\pi}{4}+\frac{2\cdot2\cdot\pi}{3}=\frac{3\pi}{12}+\frac{16\pi}{12}=\frac{19\pi}{12}$ or $\alpha=-\frac{\pi}{4}+\frac{2\cdot2\cdot\pi}{3}=-\frac{3\pi}{12}+\frac{16\pi}{12}=\frac{13\pi}{12}$

Both $\frac{19\pi}{12}$ and $\frac{13\pi}{12}$ are in the given interval.

$k = 3$: $\alpha = \frac{\pi}{4} + \frac{2 \cdot 3 \cdot \pi}{3}$ or $\alpha = -\frac{\pi}{8} + \frac{2 \cdot 3 \cdot \pi}{3}$

$= \frac{\pi}{4} + \frac{8\pi}{4}$ $\quad$ $= -\frac{\pi}{4} + \frac{8\pi}{4}$

$= \frac{9\pi}{4} > 2\pi$ $\quad$ $= \frac{7\pi}{4}$

Only $\frac{7\pi}{4}$ is in the given interval.

$k = 4$: $\alpha = \frac{\pi}{4} + \frac{2 \cdot 4 \cdot \pi}{3}$ or $\alpha = -\frac{\pi}{4} + \frac{2 \cdot 4 \cdot \pi}{3}$

$= \frac{3\pi}{12} + \frac{32\pi}{12}$ $\quad$ $= -\frac{3\pi}{12} + \frac{32\pi}{12}$

$= \frac{35\pi}{12} > 2\pi$ $\quad$ $= \frac{29\pi}{12} > 2\pi$

Since both of these values are greater than 2π, they aren't in the interval $[0, 2\pi)$. We could try larger values of k but it should be clear that since $k = 4$ resulted in a values of α that are too larger, larger values of k will produces even larger values of α so they won't produce solutions in the given interval so there's no need in trying larger values of k.

Thus, the solution set to $6\sqrt{2}\cos(3\alpha) + 10 = 4$ on the interval $[0, 2\pi)$ is $\left\{\frac{\pi}{4}, \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}, \frac{7\pi}{4}\right\}$.

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