Section V: Quadratic Equations and Functions

Module 2: The Quadratic Formula

You should remember from your course on introductory algebra that you can use the quadratic formula to solve quadratic equations.

The Quadratic Formula:

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

EXAMPLE: Solve $2x^2 + 5x - 10 = 0$ using the quadratic formula.

SOLUTION: It might be helpful to note that $a = 2$, $b = 5$, and $c = -10$ before we use the quadratic formula.

$$\begin{aligned} x &= \frac{-5 \pm \sqrt{5^2 - 4 \cdot 2 \cdot (-10)}}{2 \cdot 2} \\ &= \frac{-5 \pm \sqrt{25 + 80}}{4} \\ &= \frac{-5 \pm \sqrt{105}}{4} \end{aligned}$$

Thus, the solutions are $x = \dfrac{-5 + \sqrt{105}}{4}$ or $x = \dfrac{-5 - \sqrt{105}}{4}$, so the solution set is $\left\{ \dfrac{-5 + \sqrt{105}}{4}, \dfrac{-5 - \sqrt{105}}{4} \right\}$.

EXAMPLE: Solve $2 + \dfrac{5}{x^2} = \dfrac{9}{x}$.

SOLUTION: This isn't a quadratic equation (in fact, it is a rational equation). But if we clear the fractions by multiplying both sides of the equation be the least common denominator (which is x^2) we will obtain a quadratic equation:

$$2 + \frac{5}{x^2} = \frac{9}{x}$$

$$\Rightarrow \quad x^2 \cdot \left(2 + \frac{5}{x^2}\right) = x^2 \cdot \left(\frac{9}{x}\right)$$

$$\Rightarrow \quad 2x^2 + 5 = 9x$$

$$\Rightarrow \quad 2x^2 - 9x + 5 = 0$$

Now, we can use the quadratic formula:

$$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(2)(5)}}{2(2)}$$

$$= \frac{9 \pm \sqrt{81 - 40}}{4}$$

$$= \frac{9 \pm \sqrt{41}}{4}$$

Thus, the solution set is $\left\{\frac{9 + \sqrt{41}}{4}, \frac{9 - \sqrt{41}}{4}\right\}$.

Try this one yourself and check your answer.

Use the quadratic formula to solve the equation $2x^2 = 5x - 7$.

SOLUTION:

$$2x^2 = 5x - 7$$

$$\Rightarrow \quad 2x^2 - 5x + 7 = 0 \qquad \text{(rewrite in standard form).}$$

$$\Rightarrow \quad x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(7)}}{2(2)} \qquad \text{(use the quadratic formula)}$$

$$= \frac{5 \pm \sqrt{25 - 56}}{4}$$

$$= \frac{5 \pm \sqrt{-31}}{4}$$

$$= \frac{5 \pm i\sqrt{31}}{4} \qquad \text{(remember } \sqrt{-1} = i\text{)}$$

Thus, the solution set is $\left\{\frac{5 + i\sqrt{31}}{4}, \frac{5 - i\sqrt{31}}{4}\right\}$.

EXAMPLE: Solve $x^2 + 3x + 4 = 0$ using the quadratic formula. [Note: We solved this equation by completing-the-square in the previous module.]

SOLUTION: It might be helpful to note that $a = 1$, $b = 3$, and $c = 4$ before we use the quadratic formula.

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot 4}}{2 \cdot 1} \\ &= \frac{-3 \pm \sqrt{9 - 16}}{2} \\ &= \frac{-3 \pm \sqrt{-7}}{2} \\ &= \frac{-3 \pm i\sqrt{7}}{2} \end{aligned}$$

Thus, the solution set is $\left\{ \dfrac{-3 + i\sqrt{7}}{2}, \ \dfrac{-3 - i\sqrt{7}}{2} \right\}$.

Since the radicand is negative in the example above, the solutions to the quadratic equation are complex numbers. The radicand in the quadratic formula is called the discriminant.

DEFINITION: The **discriminant** of the quadratic equation $ax^2 + bx + c = 0$ is $b^2 - 4ac$.

EXAMPLE: The discriminant of the quadratic equation $5x^2 - 3x + 2 = 0$ is

$$\begin{aligned} b^2 - 4ac &= (-3)^2 - 4 \cdot 5 \cdot 2 \\ &= 9 - 40 \\ &= -31 \end{aligned}$$

The discriminant tells us the nature of the solutions to any quadratic equation; see the table below.

Table 1: The Discriminant

If the discriminant is…	…then there…
…positive and a perfect square	…are two rational solutions
…positive and not a perfect square	…are two irrational solutions
…zero	…is one rational solution
…negative	…are two complex solutions

EXAMPLE: Describe the nature of the solutions to the quadratic equations based on their discriminants.

a. $2x^2 = 5x - 1$

b. $x^2 - 4x = -5$

c. $3x^2 - 6x + 3 = 0$

d. $4x^2 + 7x = 2$

SOLUTION:

a. $2x^2 = 5x - 1 \;\Rightarrow\; 2x^2 - 5x + 1 = 0$

So the discriminant is $(-5)^2 - 4 \cdot 2 \cdot 1 = 17$. Since the discriminant is positive and not a perfect square, there are two irrational solutions.

b. $x^2 - 4x = -5 \;\Rightarrow\; x^2 - 4x + 5 = 0$

So the discriminant is $(-4)^2 - 4 \cdot 1 \cdot 5 = -4$. Since the discriminant is negative, there are two complex solutions.

c. $3x^2 - 6x + 3 = 0$

The discriminant is $(-6)^2 - 4 \cdot 3 \cdot 3 = 0$. Since the discriminant is zero, there is one rational solution.

d. $4x^2 + 7x = 2 \;\Rightarrow\; 4x^2 + 7x - 2 = 0$

So the discriminant is $7^2 - 4 \cdot 4(-2) = 81$. Since the discriminant is positive and a perfect square, there are two rational solutions.