Module 1: Intro. to Radical Expressions and Functions

The term **radical** is a fancy mathematical term for the things like *square roots* and *cube roots* that you may have studied in previous mathematics courses.

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**SQUARE ROOTS**

**DEFINITION:** A **square root** of a number $a$ is a number $c$ satisfying the equation $c^2 = a$.

**EXAMPLE:** A square root of 9 is 3 since $3^2 = 9$. Another square root of 9 is $-3$ since $(-3)^2 = 9$.

**DEFINITION:** The **principal square root** of a number $a$ is the nonnegative real-number square root of $a$.

**RADICAL NOTATION:** The principal square root of $a$ is denoted by $\sqrt{a}$. The symbol $\sqrt{\text{ }}$ is called a *radical sign*. The expression under the radical sign is called the *radicand*.

**EXAMPLE:** The square roots of 100 are 10 and $-10$. The principal square root of 100 is 10, which can be expressed in radical notation by the equation $\sqrt{100} = 10$.

**EXAMPLE:** Are there any real-number square roots of $-25$? According to the definition of square root (above) a square root of $-25$ would need to be a solution to the equation $c^2 = -25$. But there is no real number which when squared is negative! Thus, there is no real-number solution to this equation, so there are no real numbers that are square roots of $-25$. In fact there are no real-number square roots of ANY negative number!
Important Facts About Square Roots

1. Every positive real number has exactly TWO real-number square roots. (The two square roots of \( \alpha \) are \( \sqrt{\alpha} \) and \( -\sqrt{\alpha} \).)

2. Zero has only ONE square root: itself. \( \sqrt{0} = 0 \).

3. NO negative real number has a real-number square root.

**EXAMPLE:** Simplify the following expressions:

a. \( \sqrt{64} = 8 \)

b. \( \sqrt{\frac{9}{25}} = \frac{3}{5} \)

c. \( \sqrt{4m^2} = 2|m| \)

Here we need to use the absolute value since \( m \) could represent a negative number, but once \( m \) is squared and then “square rooted,” the result will be positive.

d. \( \sqrt{x^2 + 6x + 9} = \sqrt{(x + 3)^2} = |x + 3| \)

Again, we need the absolute value since \( x + 3 \) could represent a negative number.
The principal square root can be used to define the square root function: \( f(x) = \sqrt{x} \).

Since negative numbers don’t have square roots, the domain of the square root function is the set of non-negative real numbers: \([0, \infty)\). Let’s look at a graph of the square root function. We’ll use a table-of-values to obtain ordered pairs to plot on our graph:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = \sqrt{x} )</th>
<th>((x, f(x)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>(1, 1)</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>(4, 2)</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>(9, 3)</td>
</tr>
<tr>
<td>16</td>
<td>4</td>
<td>(16, 4)</td>
</tr>
</tbody>
</table>

The graph of \( f(x) = \sqrt{x} \).

Notice that the range of the square root function is the set of non-negative real numbers: \([0, \infty)\).

CUBE ROOTS

**DEFINITION:** The cube root of a number \( a \) is a number \( c \) satisfying the equation \( c^3 = a \).

**EXAMPLE:** The cube root of 8 is 2 since \( 2^3 = 8 \). Note that 2 is the only cube root of 8.

**RADICAL NOTATION:** The cube root of \( a \) is denoted by \( ³\sqrt{a} \).
EXAMPLE: a. The cube root of 1000 is 10 since \(10^3 = 1000\). Using radical notation, we could write \(\sqrt[3]{1000} = 10\).

b. The cube root of \(-1000\) is \(-10\) since \((-10)^3 = -1000\). Using radical notation, we could write \(\sqrt[3]{-1000} = -10\).

Important Fact About Cube Roots

Every real number has exactly ONE real-number cube root.

EXAMPLE: Simplify the following expressions.

a. \(\sqrt[3]{-512}\)

b. \(\sqrt[3]{\frac{27}{125}}\)

c. \(\sqrt[3]{8k^3}\)

d. \(\sqrt[3]{-8k^3}\)

SOLUTIONS:

a. \(\sqrt[3]{-512} = \sqrt[3]{(-8)^3} = -8\)

b. \(\sqrt[3]{\frac{27}{125}} = \sqrt[3]{\left(\frac{3}{5}\right)^3} = \frac{3}{5}\)

c. \(\sqrt[3]{8k^3} = \sqrt[3]{(2k)^3} = 2k\)
The cube root can be used to define the **cube root function**:

\[ g(x) = \sqrt[3]{x} \].

Since all real numbers have a real-number cube root, the *domain* of the cube root function is the set of real numbers, \( \mathbb{R} \). Let’s look at a graph of the cube root function.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( g(x) = \sqrt[3]{x} )</th>
<th>((x, g(x)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>-8</td>
<td>-2</td>
<td>(-8, -2)</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>(-1, -1)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>(1, 1)</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>(8, 2)</td>
</tr>
<tr>
<td>27</td>
<td>3</td>
<td>(27, 3)</td>
</tr>
</tbody>
</table>

Notice that the *range* of the cube root function is the set of real numbers, \( \mathbb{R} \).

We can make a variety of functions using the square and cube roots.
EXAMPLE: Let \( w(x) = \sqrt{2x - 5} \).

a. Evaluate \( w(3) \).

b. Evaluate \( w(15) \).

c. Evaluate \( w(0) \).

d. What is the domain of \( w \)?

SOLUTIONS:

a. \[ w(3) = \sqrt{2(3) - 5} \]
   \[ = \sqrt{6 - 5} \]
   \[ = \sqrt{1} \]
   \[ = 1 \]

b. \[ w(15) = \sqrt{2(15) - 5} \]
   \[ = \sqrt{30 - 5} \]
   \[ = \sqrt{25} \]
   \[ = 5 \]

c. \[ w(0) = \sqrt{2(0) - 5} \]
   \[ = \sqrt{0 - 5} \]
   \[ = \sqrt{-5} \]

Since there is no real number that is the square root of \(-5\), we say that \( w(0) \) does not exist.

d. Since only non-negative numbers have real-number square roots, we can only input into the function \( w \) \( x \)-values that make the expression under the square root sign non-negative, i.e., \( x \)-values that make \( 2x - 5 \geq 0 \).

\[
2x - 5 \geq 0
\]
\[
\Rightarrow \quad 2x \geq 5
\]
\[
\Rightarrow \quad x \geq \frac{5}{2}
\]

Thus, the domain of \( w \) is the set of real numbers greater than or equal to \( \frac{5}{2} \). In interval notation, the domain of \( w \) is \( \left[ \frac{5}{2}, \infty \right) \).
EXAMPLE: Let $h(t) = \sqrt[3]{t + 27}$.

a. Evaluate $h(t)$ if $t = 37$.

b. Evaluate $h(t)$ if $t = -152$.

c. Evaluate $h(t)$ if $t = 0$.

d. What is the domain of $h$?

SOLUTIONS:

a. $h(37) = \sqrt[3]{37 + 27}$
   
   $= \sqrt[3]{64}$
   
   $= 4$

b. $h(-152) = \sqrt[3]{-152 + 27}$

   $= \sqrt[3]{-125}$

   $= -5$

c. $h(0) = \sqrt[3]{0 + 27}$

   $= \sqrt[3]{27}$

   $= 3$

d. Since every real number has a cube root, there are no restrictions on which $t$-values that can be input into the function $h$. Therefore, the domain of $h$ is the set of real numbers, $\mathbb{R}$. 
OTHER ROOTS

We can extend the concept of square and cube roots and define roots based on any positive integer \( n \).

**DEFINITION:** For any integer \( n \), an \( n^{\text{th}} \) root of a number \( a \) is a number \( c \) satisfying the equation \( c^n = a \).

**RADICAL NOTATION:** The principal \( n^{\text{th}} \) root of \( a \) is denoted by \( \sqrt[n]{a} \).

**EXAMPLE:** What is the real-number 4\(^{\text{th}}\) root of 81?

**SOLUTION:** Since \( 3^4 = 81 \) and \((-3)^4 = 81\) both 3 and –3 are 4\(^{\text{th}}\) roots of 81. The principal 4\(^{\text{th}}\) root of 81 is 3 (since principal roots are positive). We can write \( \sqrt[4]{81} = 3 \).

**EXAMPLE:** What is the real-number 5\(^{\text{th}}\) root of 32?

**SOLUTION:** Since \( 2^5 = 32 \) the only 5\(^{\text{th}}\) root of 32 is 2. The principal 5\(^{\text{th}}\) root of 32 is 2 (since 2 is the only 5\(^{\text{th}}\) root of 32). We can write \( \sqrt[5]{32} = 2 \).

The two examples above expose a fundamental difference between odd and even roots. We only found one real number 5\(^{\text{th}}\) root of 32, and 5 is an odd number, but we found two real number 4\(^{\text{th}}\) roots of 81 and 4 is an even number.

**Important Facts About Odd and Even Roots**

1. Every real number has exactly ONE real-number \( n^{\text{th}} \) root if \( n \) is odd.

2. Every positive real number has TWO real-number \( n^{\text{th}} \) roots if \( n \) is even.

**NOTE:** Negative numbers do not have real-number even roots. So if \( n \) is even, we say that the \( n^{\text{th}} \) root of a negative number does not exist.
EXAMPLE: Simplify the following expressions.

a. \( \sqrt[6]{x^6} \)

b. \( \sqrt[11]{t^{11}} \)

SOLUTIONS:

a. \( \sqrt[6]{x^6} = |x| \)

Here we need to use the absolute value since \( x \) could represent a negative number but once it is raised to an even power, the result will be positive.

b. \( \sqrt[11]{t^{11}} = t \)

Here we do not need to use the absolute value since if \( t \) is negative, once it is raised to an odd power the result will still be negative, and there is a real-number 11th root of a negative number.

We can use \( n^{\text{th}} \) roots to define functions.

EXAMPLE: Let \( p(x) = \sqrt[4]{x - 1} \).

a. Evaluate \( p(17) \).

b. Evaluate \( p(-15) \).

c. Evaluate \( p(2) \).

d. What is the domain of \( p \) ?

SOLUTIONS:

a. \( p(17) = \sqrt[4]{17 - 1} \)

\[ = \sqrt[4]{16} \]

\[ = \sqrt[4]{2^4} \]

\[ = 2 \]
b. \( p(-15) = \sqrt[4]{-15} - 1 \)
\[ = \sqrt[4]{-16} \]

Since there is no real-number 4\(^{th}\) root of a negative number, we say that \( p(-15) \) is undefined.

c. \( p(2) = \sqrt[4]{2} - 1 \)
\[ = \sqrt[4]{1} \]
\[ = 1 \]

d. Since only non-negative numbers have real-number 4\(^{th}\) roots, we can only input into the function \( x \)-values that make the expression under the radical sign non-negative, i.e., \( x \)-values that make \( x - 1 \geq 0 \).
\[
\begin{align*}
 x - 1 & \geq 0 \\
 \Rightarrow \quad & x \geq 1
\end{align*}
\]

Thus, the domain of \( p \) is \([1, \infty)\).