

DEFINITION: A **rational function** is a ratio of polynomial functions. Thus, if \( p \) and \( q \) are polynomial functions, then \( r(x) = \frac{p(x)}{q(x)} \) is a rational function. (Note that \( q(x) \neq 0 \).)

**EXAMPLE:** The four functions given below are all rational functions.

\[
\begin{align*}
a. \quad \alpha(x) &= \frac{x - 5}{x^2 - 8x + 7} \\
b. \quad \beta(x) &= \frac{x^8 + 7x^5 + x - 20}{6x^7 - 15x^6 - 36x^5} \\
c. \quad \varphi(x) &= \frac{x^2 + x + 1}{x^3 + x^2 + x + 1} \\
d. \quad \delta(x) &= \frac{5}{12x^4 - 17}
\end{align*}
\]

Since rational functions involve division, we need to be sure not to divide by zero. In order to avoid division by zero, we need to exclude from the domain of a rational function any numbers that make the denominator zero.

**EXAMPLE:** What is the domain of the function \( \alpha(x) = \frac{x - 5}{x^2 - 8x + 7} \)?

**SOLUTION:**

In order to find the domain, we need to determine which numbers make the denominator of \( \alpha \) zero. So we need to solve \( x^2 - 8x - 7 = 0 \):

\[
\begin{align*}
x^2 - 8x + 7 &= 0 \\
\Rightarrow (x - 7)(x - 1) &= 0 \\
\Rightarrow x &= 7 \text{ or } x = 1
\end{align*}
\]

Since both 7 and 1 make the denominator zero, we must exclude them from the domain of \( \alpha \). Thus, the domain of \( \alpha \) is \( \{ x \in \mathbb{R} \text{ and } x \neq 7 \text{ and } x \neq 1 \} \), i.e., all real numbers except 7 and 1.
In the next module we will study what happens to the graph of a rational function near the values excluded from its domain. (Although 7 is not in the domain of the function 
\[ \alpha(x) = \frac{x - 5}{x^2 - 8x + 7} \] values like 7.01 and 6.99 are valid inputs. So the inputs can get extremely close to 7, but the input cannot equal 7. In the next module we’ll study what happens to the graph of \( \alpha \) as \( x \) get closer and closer to 7.) Before we study this, we will investigate the long-run behavior of rational functions.

### The Long-Run Behavior of Rational Functions

The long-run behavior of rational functions refers to what happens to the graph of the function as the \( x \)-values get really big or really small (which we can denote by the symbols \( x \to \pm\infty \)). Since a rational function is a ratio of polynomial functions, we can use what we learned about polynomial functions here. Recall that the long-run behavior of a polynomial function is determined by its leading term. Thus, the long-run behavior of a rational function can be found by comparing the leading terms of the polynomials in the numerator and denominator.

**Example:** Determine the long-run behavior of the function 
\[ h(x) = \frac{2x^2 - 8}{(x - 1)^2}. \]

**Solution:**

The long-run behavior of a function concerns what happens as the inputs get extreme (i.e., \( x \to \pm\infty \)). As we learned when we studied polynomials, both the numerator and denominator are dominated by their leading terms when \( x \to \pm\infty \). Thus, as \( x \to \pm\infty \),

\[ h(x) = \frac{2x^2 - 8}{(x - 1)^2} \approx \frac{2x^2}{x^2} = 2. \]

This tells us that when our inputs are really big or really small, the outputs will be about 2, which means that the graph of \( h \) gets closer and closer to the line \( y = 2 \) as \( x \to \pm\infty \) (so in the long-run, the graph of \( h \) will look like the graph of \( y = 2 \)).

In order to decide if \( h \) approaches \( y = 2 \) from above 2 or below 2 let’s evaluate \( h \) for extreme inputs and observe if the outputs are greater than 2 or less than 2.
\[ x = 100000 \Rightarrow h(100000) = \frac{2(100000)^2 - 8}{(100000 - 1)^2} \approx 2.00004 \]

So as \( x \to \infty, \) \( h(x) > 2 \)

(so as \( x \) gets really big, \( h(x) \) approaches 2 from above 2)

\[ x = -100000 \Rightarrow h(-100000) = \frac{2(-100000)^2 - 8}{(-100000 - 1)^2} \approx 1.9999 \]

So as \( x \to -\infty, \) \( h(x) < 2 \)

(so as \( x \) gets really small, \( h(x) \) approaches 2 from below 2)

With this information, we can sketch the long-run behavior of \( h, \) see Figure 1.

\[ (Note \ that \ in \ the \ next \ module \ we \ will \ study \ the \ short \ term \ behavior \ of \ rational \ functions, \ and \ finish \ sketching \ the \ graph \ of \ h.) \]

The line \( y = 2 \) plays a crucial role in the long-term behavior of the rational function \( h. \) Thus, this line is called a horizontal asymptote. Hopefully you recall the following definition (from the Section III: Module 1: Introduction to Exponential Functions).

**DEFINITION:** A horizontal asymptote is a horizontal line that the graph of a function gets arbitrarily close to as the input values get very large (or very small).
EXAMPLE: Determine the long-run behavior of the function \( k(x) = \frac{x}{x^2 - 25} \). (What is the horizontal asymptote of \( k \)?)

SOLUTION:

To find the long-run behavior of this rational function, we need to recognize that as \( x \to \pm \infty \), the leading terms of the numerator and denominator determine the nature of the outputs:

\[
k(x) = \frac{x}{x^2 - 25} \approx \frac{x}{x^2} = \frac{1}{x}
\]

So as \( x \to \pm \infty \), the graph of \( k \) looks like \( y = \frac{1}{x} \). But we know that as \( x \to \pm \infty \), \( \frac{1}{x} \) gets closer and closer to zero. So we see that as \( x \to \pm \infty \), \( k(x) \approx 0 \), which means that \( y = 0 \) is the horizontal asymptote. To decide if \( k \) approaches \( y = 0 \) from above 0 or below 0 let’s evaluate \( k \) for extreme inputs and observe if the outputs are greater than 0 or less than 0.

\[
x = 100000 \quad \Rightarrow \quad k(100000) = \frac{100000}{(100000)^2 - 25} \approx 0.00001
\]

So as \( x \to \infty \), \( k(x) > 0 \)

(since as \( x \) gets really big, \( k(x) \) approaches 0 from above 0)

\[
x = -100000 \quad \Rightarrow \quad k(-100000) = \frac{-100000}{(-100000)^2 - 25} \approx -0.00001
\]

So as \( x \to -\infty \), \( k(x) < 0 \)

(since as \( x \) gets really small, \( k(x) \) approaches 0 from below 0)

With this information, we can sketch the long-run behavior of \( k \); see Figure 2.

![Figure 2: The long run behavior of \( y = k(x) \).](image)

(We will finish the graph of \( k \) in the next module.)
EXAMPLE: Determine the long-run behavior of the function \( m(x) = \frac{x^3 - 16x}{x^2 - 64} \).

SOLUTION:

To find the long-run behavior of this rational function, we need recognize that as \( x \to \pm \infty \), the leading terms of the numerator and denominator determine the nature of the outputs:

\[
m(x) = \frac{x^3 - 16x}{x^2 - 64} \approx \frac{x^3}{x^2} = x
\]

So as \( x \to \pm \infty \), \( m \) looks like \( y = x \). This isn’t a horizontal line, but it is an asymptote nonetheless. If an asymptote is a non-horizontal line, it is often called **oblique**. Thus, \( y = x \) is an **oblique asymptote** for the function \( m \). In order to sketch the long-run behavior of \( m \) we need to decide if the graph of \( m \) approaches its oblique asymptote \( y = x \) from above or below:

\[
x = 100000 \quad \Rightarrow \quad m(100000) = \frac{(100000)^3 - 16}{(100000)^2 - 64} \approx 100000.000486 > 100000
\]

So as \( x \to \infty \), \( m(x) > x \)

(so as \( x \) gets really big, \( m(x) \) approaches \( y = x \) from above \( y = x \))

\[
x = -100000 \quad \Rightarrow \quad m(-100000) = \frac{(-100000)^3 - 16}{(-100000)^2 - 64} \approx -100000.000486 < -100000
\]

So as \( x \to -\infty \), \( m(x) < x \)

(so as \( x \) gets really small, \( m(x) \) approaches \( y = x \) from below \( y = x \))

With this information, we can sketch the long-run behavior of \( m \); see Figure 3.

![Figure 3](image_url)

**Figure 3:** The graph of \( y = x \) and the long-term behavior of \( y = m(x) \).

(We will finish the graph of \( m \) in the next module.)