DEFINITION: A power function is a function of the form \( f(x) = kx^p \) where \( k \) and \( p \) are constants.

EXAMPLE: Which of the following functions are power functions? For each power function, state the value of the constants \( k \) and \( p \) in the formula \( y = kx^p \).

\[
\begin{align*}
\text{a. } b(x) &= 5(x - 3)^4 \\
\text{b. } m(x) &= 7 \sqrt[4]{x} \\
\text{c. } l(x) &= 3 \cdot 2^x \\
\text{d. } s(x) &= \sqrt[5]{\frac{7}{x}}
\end{align*}
\]

SOLUTIONS:

\[
\begin{align*}
\text{a. } &\text{ The function } b(x) = 5(x - 3)^4 \text{ is not a power function because we cannot write it in the form } y = kx^p. \\
\text{b. } &\text{ The function } m(x) = 7 \sqrt[4]{x} \text{ is a power function because we can rewrite its formula as } m(x) = 7 \cdot x^{1/4}. \text{ So } k = 7 \text{ and } p = \frac{1}{4}. \\
\text{c. } &\text{ The function } l(x) = 3 \cdot 2^x \text{ is not a power function because the power is not constant. In fact, } l(x) = 3 \cdot 2^x \text{ is an exponential function.} \\
\text{d. } &\text{ Since } \\
&\sqrt[5]{\frac{7}{x^5}} = \frac{\sqrt[5]{7}}{x^{5/2}} \\
&= \sqrt[5]{7} \cdot x^{-5/2}
\end{align*}
\]

we see that \( s(x) = \sqrt[5]{\frac{7}{x^5}} \) can be written in the form \( y = kx^p \) where \( k = \sqrt[5]{7} \) and \( p = -\frac{5}{2} \), so \( s \) is a power function.
As is the case with linear functions and exponential functions, given two points on the graph of a power function, we can find the function’s formula.

**EXAMPLE:** Suppose that the points (1, 81) and (3, 729) are on the graph of a function \( f \). Find an algebraic rule for \( f \) assuming that it is ...

a. a linear function. 

b. an exponential function.

c. a power function.

**SOLUTIONS:**

a. If \( f \) is a linear function we know that its rule has form \( f(x) = mx + b \). We can use the two given points to solve for \( m \).

\[
m = \frac{729 - 81}{3 - 1} = \frac{648}{2} = 324
\]

So now we know that \( f(x) = 324x + b \). We can use either one of the given points to find \( b \). Let’s use (1, 81):

\[
(1, 81) \Rightarrow f(1) = 81 = 324(1) + b \\
\Rightarrow b = 81 - 324 \\
\Rightarrow b = -243
\]

Thus, if \( f \) is linear, its rule is \( f(x) = 324x - 243 \).

b. If \( f \) is an exponential function we know its rule has form \( f(x) = ab^x \). We can use the two given points to find two equations involving \( a \) and \( b \):

\[
(1, 81) \Rightarrow f(1) = 81 = ab^1 \\
(3, 729) \Rightarrow f(3) = 729 = ab^3.
\]

In Section III: Module 2 we solved similar systems of equations by forming ratios. Let’s try a different method here: the substitution method.

Let’s start by solving the first equation for \( a \):

\[
81 = ab^1 \\
\Rightarrow a = \frac{81}{b}
\]
Now we can substitute the expression $\frac{81}{b}$ for $a$ in the second equation:

$$
729 = ab^3
\Rightarrow 729 = \frac{81}{b} \cdot b^3
\Rightarrow 729 = 81 \cdot b^2
\Rightarrow \frac{729}{81} = b^2
\Rightarrow 9 = b^2
\Rightarrow b = \sqrt{9} = 3 \quad \text{(we don't need } \pm \sqrt{9} \text{ since the base of an exponential function is always positive)}
$$

Now that we know what $b$ is, we can use the fact that $a = \frac{81}{b}$ to find $a$:

$$
\begin{align*}
a &= \frac{81}{b} \\
&= \frac{81}{3} \\
&= 27
\end{align*}
$$

Thus, if $f$ is exponential, its rule is $f(x) = 27 \cdot 3^x$.

c. Since $f$ is a power function we know that its rule has form $f(x) = kx^p$. We can use the two given points to find two equations involving $k$ and $p$:

$$
\begin{align*}
(1, 81) & \Rightarrow f(1) = 81 = k(1)^p \\
(3, 729) & \Rightarrow f(3) = 729 = k(3)^p.
\end{align*}
$$

We can use the first equation to immediately find $k$.

$$
81 = k(1)^p \\
\Rightarrow k = 81
$$

Now we can find $p$ by substituting $k = 81$ into the second equation:

$$
\begin{align*}
729 &= 81(3)^p \\
\Rightarrow \frac{729}{81} &= 3^p \\
\Rightarrow 9 &= 3^p \quad \text{(note that this could be solved with logarithms if the solution weren't so obvious)} \\
\Rightarrow p &= 2
\end{align*}
$$

Thus, if $f$ is a power function, its rule is $f(x) = 81x^2$. 

Graphs of Power Functions

For a power function \( y = kx^p \) the greater the power of \( p \), the faster the outputs grow. Below are the graphs of six power functions. Notice that as the power increases, the outputs increase more and more quickly. As \( x \) increases without bound (written \( x \to \infty \)), higher powers of \( x \) get a lot larger than (i.e., dominate) lower powers of \( x \). (Note that we are discussing the **long-term** behavior of the function.)

\[
\begin{align*}
y &= x, \quad y = x^{3/2}, \quad y = x^2, \quad y = x^3, \quad y = x^4, \quad y = x^5.
\end{align*}
\]

As \( x \) approaches zero (written "\( x \to 0 \)"), the story is completely different. If \( x \) is between 0 and 1, \( x^3 \) is larger than \( x^4 \), which is larger than \( x^5 \). (Try \( x = 0.1 \) to confirm this). For values of \( x \) near zero, smaller powers dominate. On the graph below, notice how on the interval \((0, 1)\) the linear power function \( y = x \) dominates power functions of larger power.

\[
\begin{align*}
y &= x, \quad y = x^{3/2}, \quad y = x^2, \quad y = x^3, \quad y = x^4, \quad y = x^5.
\end{align*}
\]
EXAMPLE: Use your graphing calculator to graph \( f(x) = 1000x^3 \) and \( g(x) = x^4 \) for \( x > 0 \). Compare the long-term behavior of these two functions.

CLICK HERE (Be sure to turn up the volume on your computer!)

Could the graphs of \( f(x) = 1000x^3 \) and \( g(x) = x^4 \) intersect again for some value of \( x > 1000 \)? To determine where these graphs intersect, let’s solve the equation \( f(x) = g(x) \):

\[
f(x) = g(x)
\]

\[
1000x^3 = x^4
\]

\[
0 = x^4 - 1000x^3
\]

\[
0 = x^3(x - 1000).
\]

Since the only solutions to this equation are \( x = 0 \) and \( x = 1000 \), the graphs of \( f(x) = 1000x^3 \) and \( g(x) = x^4 \) only intersect at \( x = 0 \) and \( x = 1000 \), so they do not intersect when \( x > 1000 \).

EXAMPLE: Use your graphing calculator to graph the power function \( f(x) = x^3 \) and the exponential function \( g(x) = 2^x \) for \( x > 0 \). Compare the long-term behavior of these two functions.

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Key Point: Any positive increasing exponential function eventually grows faster than any power function.