8.8 Applications of Taylor Polynomials

Infinite Sequences and Series **8**

Applications of Taylor Polynomials

In this section we explore two types of applications of Taylor polynomials. First we look at how they are used to approximate functions—computer scientists like them because polynomials are the simplest of functions.

Then we investigate how physicists and engineers use them in such fields as relativity, optics, blackbody radiation, electric dipoles, and building highways across a desert.

Approximating Functions by Polynomials

Suppose that $f(x)$ is equal to the sum of its Taylor series at a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

We have introduced the notation $T_n(x)$ for the nth partial sum of this series and called it the nth-degree Taylor polynomial of f at a. Thus

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Approximating Functions by Polynomials

Since f is the sum of its Taylor series, we know that $T_n(x) \to f(x)$ as n $\to \infty$ and so T_n can be used as an approximation to f: $f(x) \approx T_n(x)$.

Notice that the first-degree Taylor polynomial

$$T_1(x) = f(a) + f'(a)(x - a)$$

is the same as the linearization of f at a.

Notice also that T_1 and its derivative have the same values at a that f and f' have. In general, it can be shown that the derivatives of T_n at a agree with those of f up to and including derivatives of order n.

Approximating Functions by Polynomials

To illustrate these ideas let's take another look at the graphs of $y = e^x$ and its first few Taylor polynomials, as shown in Figure 1.

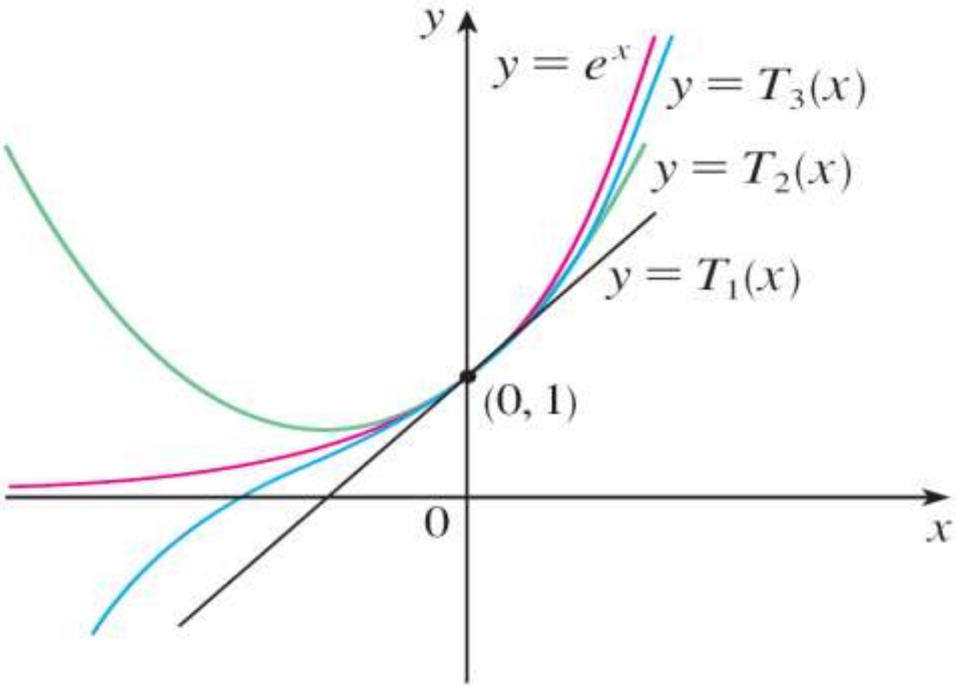

Figure 1

Approximating Functions by Polynomials

The graph of T_1 is the tangent line to $y = e^x$ at (0, 1); this tangent line is the best linear approximation to e^x near (0, 1).

The graph of T_2 is the parabola $y = 1 + x + x^2/2$, and the graph of T_3 is the cubic curve $y = 1 + x + x^2/2 + x^3/6$, which is a closer fit to the exponential curve $y = e^x$ than T_2.

The next Taylor polynomial T_4 would be an even better approximation, and so on.

Approximating Functions by Polynomials

The values in the table give a numerical demonstration of the convergence of the Taylor polynomials $T_n(x)$ to the function $y = e^x$.

	$x = 0.2$	$x = 3.0$
$T_2(x)$	1.220000	8.500000
$T_4(x)$	1.221400	16.375000
$T_6(x)$	1.221403	19.412500
$T_8(x)$	1.221403	20.009152
$T_{10}(x)$	1.221403	20.079665
e^x	1.221403	20.085537

Approximating Functions by Polynomials

We see that when $x = 0.2$ the convergence is very rapid, but when $x = 3$ it is somewhat slower. In fact, the farther x is from 0, the more slowly $T_n(x)$ converges to e^x.

When using a Taylor polynomial T_n to approximate a function f, we have to ask the questions: How good an approximation is it? How large should we take n to be in order to achieve a desired accuracy? To answer these questions we need to look at the absolute value of the remainder:

$$|R_n(x)| = |f(x) - T_n(x)|$$

Approximating Functions by Polynomials

There are three possible methods for estimating the size of the error:

1. If a graphing device is available, we can use it to graph $|R_n(x)|$ and thereby estimate the error.

2. If the series happens to be an alternating series, we can use the Alternating Series Estimation Theorem.

3. In all cases we can use Taylor's Inequality, which says that if $|f^{(n+1)}(x)| \leq M$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

Example 1 – *Approximating a Root Function by a Quadratic Function*

(a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.

(b) How accurate is this approximation when $7 \le x \le 9$?

Solution:

(a)

$$f(x) = \sqrt[3]{x} = x^{1/3} \qquad f(8) = 2$$

$$f'(x) = \tfrac{1}{3}x^{-2/3} \qquad f'(8) = \tfrac{1}{12}$$

$$f''(x) = -\tfrac{2}{9}x^{-5/3} \qquad f''(8) = -\tfrac{1}{144}$$

$$f'''(x) = \tfrac{10}{27}x^{-8/3}$$

Example 1 – *Solution*

cont'd

Thus the second-degree Taylor polynomial is

$$T_2(x) = f(8) + \frac{f'(8)}{1!}(x - 8) + \frac{f''(8)}{2!}(x - 8)^2$$

$$= 2 + \tfrac{1}{12}(x - 8) - \tfrac{1}{288}(x - 8)^2$$

The desired approximation is

$$\sqrt[3]{x} \approx T_2(x)$$

$$= 2 + \tfrac{1}{12}(x - 8) - \tfrac{1}{288}(x - 8)^2$$

Example 1 – *Solution*

cont'd

(b) The Taylor series is not alternating when $x < 8$, so we can't use the Alternating Series Estimation Theorem in this example.

But we can use Taylor's Inequality with $n = 2$ and $a = 8$:

$$|R_2(x)| \leqslant \frac{M}{3!}|x - 8|^3$$

where $|f'''(x)| \leq M$.

Because $x \geq 7$, we have $x^{8/3} \geq 7^{8/3}$ and so

$$f'''(x) = \frac{10}{27} \cdot \frac{1}{x^{8/3}} \leqslant \frac{10}{27} \cdot \frac{1}{7^{8/3}} < 0.0021$$

Therefore we can take $M = 0.0021$.

Example 1 – *Solution*

cont'd

Also $7 \le x \le 9$, so $-1 \le x - 8 \le 1$ and $|x - 8| \le 1$.

Then Taylor's Inequality gives

$$|R_2(x)| \leqslant \frac{0.0021}{3!} \cdot 1^3 = \frac{0.0021}{6} < 0.0004$$

Thus, if $7 \le x \le 9$, the approximation in part (a) is accurate to within 0.0004.

Approximating Functions by Polynomials

Figure 6 shows the graphs of the Maclaurin polynomial approximations

$$T_1(x) = x \qquad T_3(x) = x - \frac{x^3}{3!}$$

$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \qquad T_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

to the sine curve.

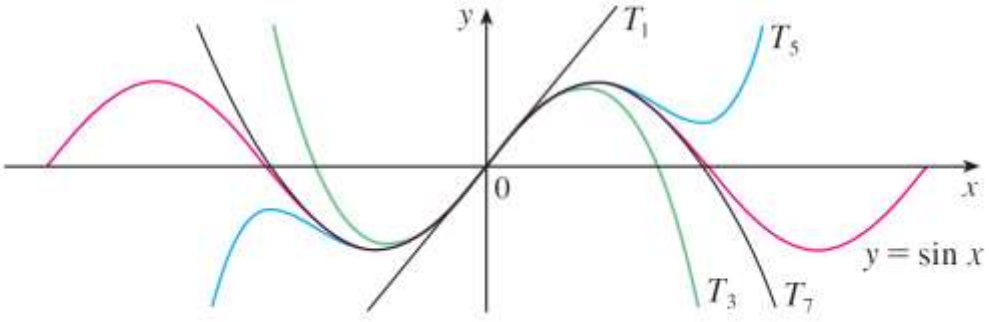

Figure 6

Approximating Functions by Polynomials

You can see that as n increases, $T_n(x)$ is a good approximation to $\sin(x)$ on a larger and larger interval.

One use of the type of calculation done in Examples 1 occurs in calculators and computers.

For instance, when you press the sin or e^x key on your calculator, or when a computer programmer uses a subroutine for a trigonometric or exponential or Bessel function, in many machines a polynomial approximation is calculated.

Applications to Physics

Taylor polynomials are also used frequently in physics. In order to gain insight into an equation, a physicist often simplifies a function by considering only the first two or three terms in its Taylor series.

In other words, the physicist uses a Taylor polynomial as an approximation to the function. Taylor's Inequality can then be used to gauge the accuracy of the approximation.

Example 3 – *Using Taylor to Compare Einstein and Newton*

In Einstein's theory of special relativity the mass of an object moving with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_o is the mass of the object when at rest and c is the speed of light. The kinetic energy of the object is the difference between its total energy and its energy at rest:

$$K = mc^2 - m_o c^2$$

Example 3 – *Using Taylor to Compare Einstein and Newton*

(a) Show that when v is very small compared with c, this expression for K agrees with classical Newtonian physics: $K = \frac{1}{2}m_0v^2$.

(b) Use Taylor's Inequality to estimate the difference in these expressions for K when $|v| \le 100$ m/s.

Example 3 – *Using Taylor to Compare Einstein and Newton*

(a)Show that when *v* is very small compared with *c*, this expression for *K* agrees with classical Newtonian physics:

(b) Use Taylor's Inequality to estimate the difference in these expressions for *K* when $|v| \leq 100$ m/s.

Solution:

(a) Using the expressions given for *K* and *m*, we get

$$K = mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} - m_0c^2$$

$$= m_0c^2\left[\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right]$$

Example 3 – *Solution*

cont'd

With $x = -v^2/c^2$, the Maclaurin series for $(1 + x^2)^{-1/2}$ is most easily computed as a binomial series with $k = -\frac{1}{2}$. (Notice that $|x| < 1$ because $v < c$.)

Therefore we have

$$(1 + x)^{-1/2} = 1 - \tfrac{1}{2}x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}x^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}x^3 + \cdots$$

$$= 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3 + \cdots$$

and

$$K = m_0c^2\left[\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right) - 1\right]$$

$$= m_0c^2\left(\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right)$$

Example 3 – *Solution*

cont'd

If v is much smaller than c, then all terms after the first are very small when compared with the first term. If we omit them, we get

$$K \approx m_0 c^2 \left(\frac{1}{2} \frac{v^2}{c^2} \right) = \tfrac{1}{2} m_0 v^2$$

(b) If $x = -v^2/c^2$, $f(x) = m_o c^2[(1 + x)^{-1/2} - 1]$, and M is a number such that $|f''(x)| \leq M$, then we can use Taylor's Inequality to write

$$|R_1(x)| \leqslant \frac{M}{2!} x^2$$

Example 3 – *Solution*

cont'd

We have $f''(x) = \frac{3}{4} m_o c^2(1 + x)^{-5/2}$ and we are given that $|v| \leq 100$ m/s, so

$$|f''(x)| = \frac{3m_0c^2}{4(1 - v^2/c^2)^{5/2}} \leqslant \frac{3m_0c^2}{4(1 - 100^2/c^2)^{5/2}} \quad (= M)$$

Thus, with $c = 3 \times 10^8$ m/s,

$$|R_1(x)| \leqslant \frac{1}{2} \cdot \frac{3m_0c^2}{4(1 - 100^2/c^2)^{5/2}} \cdot \frac{100^4}{c^4} < (4.17 \times 10^{-10})m_0$$

So when $|v| \leq 100$ m/s, the magnitude of the error in using the Newtonian expression for kinetic energy is at most $(4.2 \times 10^{-10})m_o$.

Example 4 – *An Electric Dipole*

An electric dipole consists of two electric charges of equal magnitude and opposite sign. If the charges are q and $-q$ and are located at a distance d from each other

then the electric field E at the point P in the figure is:

$$E = \frac{q}{D^2} - \frac{q}{(D+d)^2}$$

By expanding this expression for E as a series in powers of d/D, show that E is approximately proportional to $1/D^3$ when P is far away from the dipole.

Example 4 – *An Electric Dipole*

Applications to Physics

Another application to physics occurs in optics. See figure 8.

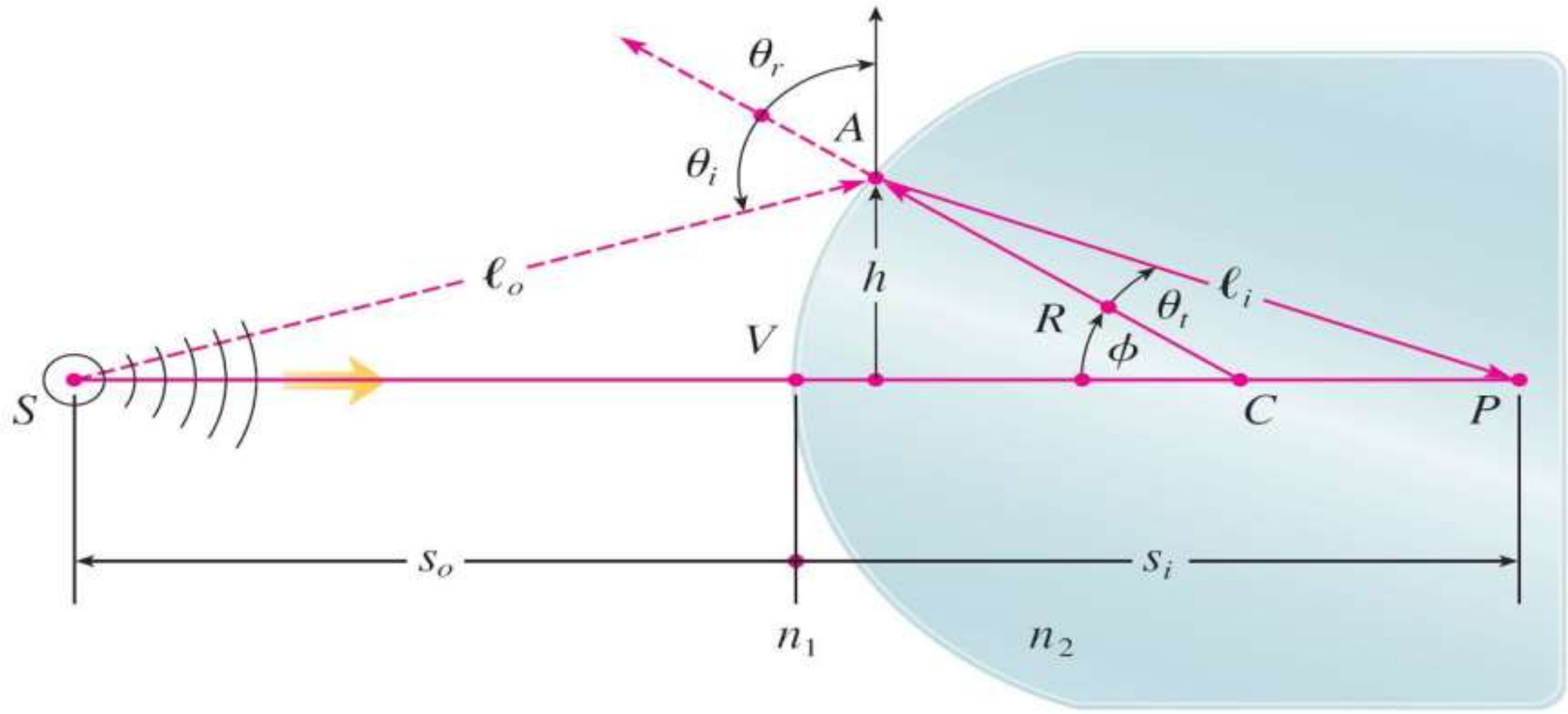

Figure 8

Refraction at a spherical interface

Applications to Physics

It depicts a wave from the point source *S* meeting a spherical interface of radius *R* centered at *C*. The ray *SA* is refracted toward *P*.

Using Fermat's principle that light travels so as to minimize the time taken, Hecht derives the equation

$$\boxed{1} \qquad \frac{n_1}{\ell_o} + \frac{n_2}{\ell_i} = \frac{1}{R}\left(\frac{n_2 s_i}{\ell_i} - \frac{n_1 s_o}{\ell_o}\right)$$

where n_1 and n_2 are indexes of refraction and l_o, l_i, s_o, and s_i are the distances indicated in Figure 8.

Applications to Physics

By the Law of Cosines, applied to triangles *ACS* and *ACP*, we have

$$\boxed{2} \qquad \ell_o = \sqrt{R^2 + (s_o + R)^2 - 2R(s_o + R)\cos\phi}$$

$$\ell_i = \sqrt{R^2 + (s_i - R)^2 + 2R(s_i - R)\cos\phi}$$

Gauss, in 1841, simplified Equation 1, by using the linear approximation cos(ϕ) ≈ 1 for small values of ϕ. (This amounts to using the Taylor polynomial of degree 1.)

Applications to Physics

Then Equation 1 becomes the following simpler equation:

$$\boxed{3} \qquad \frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R}$$

The resulting optical theory is known as *Gaussian optics,* or *first-order optics,* and has become the basic theoretical tool used to design lenses.

A more accurate theory is obtained by approximating cos(ϕ) by its Taylor polynomial of degree 3 (which is the same as the Taylor polynomial of degree 2).

Applications to Physics

This takes into account rays for which ϕ is not so small, that is, rays that strike the surface at greater distances *h* above the axis.

$$\boxed{4} \qquad \frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} + h^2\left[\frac{n_1}{2s_o}\left(\frac{1}{s_o} + \frac{1}{R}\right)^2 + \frac{n_2}{2s_i}\left(\frac{1}{R} - \frac{1}{s_i}\right)^2\right]$$

The resulting optical theory is known as *third-order optics.*